ON THE UNIVERSALITY OF INTEGRABLE DEFORMATIONS OF SOLUTIONS OF DEGENERATE RIEMANN–HILBERT–BIRKHOFF PROBLEMS

GIORDANO COTTI°

° *Faculdade de Ciências da Universidade de Lisboa*
Grupo de Física Matemática
Campo Grande Edifício C6, 1749-016 Lisboa, Portugal

Abstract. This paper addresses the classification problem of integrable deformations of solutions of "degenerate" Riemann–Hilbert–Birkhoff (RHB) problems. These consist of those RHB problems whose initial datum has diagonal pole part with coalescing eigenvalues. On the one hand, according to theorems of B. Malgrange, M. Jimbo, T. Miwa, and K. Ueno, in the non-degenerate case, there exists a universal integrable deformation inducing (via a unique map) all other deformations [JMU81, Mal83a, Mal83b, Mal86]. On the other hand, in the degenerate case, C. Sabbah proved, under sharp conditions, the existence of an integrable deformation of solutions, sharing many properties of the one constructed by Malgrange–Jimbo–Miwa–Ueno [Sab21]. Albeit the integrable deformation constructed by Sabbah is not, stricto sensu, universal, we prove that it satisfies a *relative* universal property. We show the existence and uniqueness of a maximal class of integrable deformations all induced (via a unique map) by Sabbah's integrable deformation. Furthermore, we show that such a class is large enough to include all generic integrable deformations whose pole and deformation parts are locally holomorphically diagonalizable. In itinere, we also obtain a characterization of holomorphic matrix-valued maps which are locally holomorphically Jordanizable. This extends, to the case of several complex variables, already known results independently obtained by Ph.G.A. Thijsse and W. Wasow [Thi85, Was85].

Contents

° *E-mail*: gcotti@fc.ul.pt, gcotti@sissa.it

1. Introduction

1.1. Riemann–Hilbert–Birkhoff (RHB) problems: the analytical and geometrical settings.

Consider the n-dimensional system of ordinary differential equations

$$z\frac{d}{dz}Y=\mathcal{A}(z)Y,\qquad \mathcal{A}(z)=z^r\sum_{k=0}^{\infty}\mathcal{A}_kz^{-k},\quad \mathcal{A}_0\neq 0, \tag{1.1}$$

where the series is convergent for $|z|>R_1$, and r is a non-negative integer, called the *Poincaré rank* of (1.1) at $z=\infty$. In 1909, and again in 1913, G. Birkhoff addressed the following question [Bir09, Bir13]:

Riemann–Hilbert–Birkhoff Problem (analytical version): *Does it exist an analytic matrix-valued function $T(z)=\sum_{k=0}^{\infty}T_kz^{-k}$, with $T_0\in GL(n,\mathbb{C})$ and the series converging for $|z|>R_2$, such that the transformed equation*

$$z\frac{d}{dz}Z=\widehat{\mathcal{A}}(z)Z,\qquad Y(z)=T(z)Z(z),\quad \widehat{\mathcal{A}}=T^{-1}\mathcal{A}T-zT^{-1}\frac{d}{dz}T,$$

has a polynomial matrix coefficient of the form

$$\widehat{\mathcal{A}}(z)=z^r\widehat{\mathcal{A}}_0+z^{r-1}\widehat{\mathcal{A}}_1+\dots+\widehat{\mathcal{A}}_r,\qquad \widehat{\mathcal{A}}_j\in M(n,\mathbb{C})?$$

In general, the solvability of the RHB problem is an open problem: although several[1] sufficient conditions for the solvability have been given [Bir13, JLP76, Bal90, Bol94a, Bol94b,

[1]More details on known results will be given in the main body of the paper, see Section 4.

BB97][Sab07, Ch. IV], it may happen that the RHB Problem does not always admit a positive answer (contrarily to what Birkhoff believed to have proved), see [Gan59, Mas59].

In more geometrical terms, the RHB Problem can be recast as follows:

Riemann–Hilbert–Birkhoff Problem (geometrical version): *Given a trivial vector bundle $E^{\rm in}$ on a disc $D \subseteq \mathbb{P}^1$ centered at $z = \infty$, equipped with a meromorphic connection $\nabla^{\rm in}$ with a pole at $z = \infty$, is it true that $(E^{\rm in}, \nabla^{\rm in})$ extends to a pair (E^o, ∇^o), where E^o is a* trivial *vector bundle on $\mathbb{P}^1$, and ∇^o is a meromorphic connection with only another* logarithmic *pole at $z = 0$?*

If one does not insist on the triviality of the vector bundle E^o, or on the logarithmic nature of the pole $z = 0$ of ∇^o, then the problem is easily solvable, see [Sab98, App. A.2]. What makes the RHB Problem difficult is the conjunction of these requirements.

This paper is devoted to the study of *families* of RHB problems, rather than focusing on a single one. If (X, x_o) is a pointed complex manifold, we consider families of equations (1.1), parametrized by points of X, that is

$$z\frac{d}{dz}Y = \mathcal{A}(z,x)Y, \qquad \mathcal{A}(z,x) = z^r \sum_{k=0}^{\infty} \mathcal{A}_k(x) z^{-k}, \quad \mathcal{A}_k \colon X \to M(n, \mathbb{C}) \text{ holomorphic,} \tag{1.2}$$

with the further assumptions that $\mathcal{A}_0$ is not identically zero, and that the series defining $\mathcal{A}(z,x)$ is convergent for $|z| > R$ (independent of x). The family (1.2) can be interpreted as a *deformation* of the equation given by the specialization $x = x_o$. The deformation is said to be *integrable* if there exist holomorphic matrix-valued functions $\Theta_1, \dots, \Theta_{\dim X}$ on $\{|z| > R\} \times X$ such that (1.2) is compatible with the system of equations

$$\mathrm{d}'Y = \Theta Y, \qquad \mathrm{d}' = \sum_{j=1}^{\dim X} \frac{\partial}{\partial x^j}\mathrm{d}x^j, \quad \Theta(z,x) = \sum_{j=1}^{\dim X} \Theta_j(z,x)\mathrm{d}x^j. \tag{1.3}$$

In the geometrical description, the joint system of equations (1.2),(1.3) defines a flat meromorphic connection ∇ on a trivial vector bundle E over $D \times X$, with a pole along $\{\infty\} \times X$ of order $r+1$. The connection ∇ is a *deformation* of its restriction at $x = x_o$, namely the connection $\iota^*\nabla$ on the pulled-back vector bundle ι^*E, where $\iota \colon D \to D \times X$, $z \mapsto (z, x_o)$.

From now on (including the main part of the paper), we consider[2] the case $r = 1$ only. In this case, the Birkhoff normal form reads

$$\frac{d}{dz}Z = \left(\widehat{\mathcal{A}}_0 + \frac{1}{z}\widehat{\mathcal{A}}_1\right) Z, \qquad \widehat{\mathcal{A}}_0, \widehat{\mathcal{A}}_1 \in M(n, \mathbb{C}).$$

Given a family of RHB problems, parametrized by a pointed complex manifold (X, x_o), we can raise a legitimate question: is solvability an open property? More precisely:

[2]We expect that many results mentioned (or obtained) in this paper can be generalized to higher Poincaré ranks.

Question: Assume that the RHB problem defined by the system (1.2) is solvable when specialized at $x_o \in X$. Is it true that there exists an open neighborhood $U \subseteq X$ of x_o such that the RHB problem is solvable when specialized at $x \in U$?

According to a theorem originally due to B. Malgrange (in [Mal83a, Th. 2.2.(1)] the result is formulated with some un-necessary assumptions), and subsequently refined by C. Sabbah [Sab07, Th. VI.2.1] (see also [DH21, Th. 5.1(c)]), the answer to the question above is positive if

- the deformation is integrable,
- the manifold X is simply connected[3].

Under these assumptions, one can prove the existence of a sufficiently small open neighborhood $U \subseteq X$ of x_o, and the existence and uniqueness of a frame of sections of $E|_{D\times U}$, with respect to which $\nabla|_{D\times U}$ has matrix of connections 1-forms

$$\Omega(z,x) = -\left(A(x) + \frac{1}{z}B_o\right)\mathrm{d}z - z\,C(x), \qquad x \in U, \tag{1.4}$$

where A is a holomorphic matrix-valued function on U (referred to as the *pole part* of ∇), B_o is a constant matrix, and C is a holomorphic matrix-valued 1-form on U (referred to as the *deformation part* of ∇). See Theorem 4.5.

Remark 1.1. Assume to be given a family of RHB problems as in (1.2) (for the moment not necessarily an integrable deformation), solvable for each value of the deformation parameter $x \in X$. The solutions of the RHB problems define a family of equations

$$\frac{d}{dz}Z = \left(\widehat{\mathcal{A}}_0(x) + \frac{1}{z}\widehat{\mathcal{A}}_1(x)\right)Z, \qquad \widehat{\mathcal{A}}_0, \widehat{\mathcal{A}}_1 \colon X \to M(n,\mathbb{C}) \text{ holomorphic.} \tag{1.5}$$

For each fixed $x \in X$, the solutions $Z(z,x)$ are multivalued functions, and they manifest both monodromy and a Stokes phenomenon (at $z = \infty$). The integrability condition of the deformation can be recast in terms of the family of equations (1.5) only: one can prove that the family of RHB is an integrable deformation if and only if the family of equations (1.5) is *weakly isomonodromic* in the sense of [Guz18]. This means that there exists a fundamental system of solutions $Z(z,x)$ whose monodromy matrix $M := Z(z,x)^{-1}Z(e^{2\pi\sqrt{-1}}z,x)$ does not depend on $x \in X$. Other refined sets of monodromy data (e.g. the Stokes matrices) may still depend on $x \in X$.

1.2 **Malgrange–Jimbo–Miwa–Ueno universal integrable deformations, and its Sabbah's analogue.** Consider a trivial vector bundle E^o on $\mathbb{P}^1$, equipped with a meromorphic connection ∇^o, defined (in a suitable basis of sections) by matrix Ω_o of connection 1-forms in Birkhoff normal form, namely

$$\Omega_o(z) = -\left(A_o + \frac{1}{z}B_o\right)\mathrm{d}z, \qquad A_o, B_o \in M(n,\mathbb{C}). \tag{1.6}$$

By the discussion of the previous section, the germ of an arbitrary integrable deformation (∇, E, X, x_o) of (E^o, ∇^o) can be defined by a matrix $\Omega(z,x)$ as in (1.4), where $A(x_o) = A_o$.

[3]This can always be realized up to replacing X with a simply connected neighborhood of x_o.

Denote by $\mathcal{M}_{\rm reg}$ and $\mathcal{M}_{\rm diag}$ the subsets of $M(n,\mathbb{C})$ of regular and diagonalizable $n\times n$-matrices, respectively. These sets can be defined as finite unions of smooth semi-algebraic strata in $M(n,\mathbb{C})$, called *bundles of matrices*, introduced by V.I. Arnol'd [Arn71], see Section 2. Each bundle of matrices is defined by fixing the "type" of the Jordan form of its elements, i.e. by fixing the number of Jordan blocks of each size (the numerical values of the eigenvalues is free).

As it will be explained below, the classification problem of germs of integrable deformations of ∇^o is of varying difficulty, according to which bundle the matrix $A_o\in M(n,\mathbb{C})$ belongs (i.e. to the Jordan form of A_o).

In the case $A_o\in\mathcal{M}_{\rm reg}$, the classification of germs of integrable deformations is completely understood. According to a theorem of B. Malgrange [Mal83a, Mal86], indeed, there exists a germ of *universal* integrable deformation $(\nabla^{\rm un},\underline{\mathbb{C}^n},\mathbb{C}^n,\boldsymbol{u}_o)$, where $\underline{\mathbb{C}^n}$ denotes the trivial vector bundle $\mathbb{C}^n\times\mathbb{C}^n$ over $\mathbb{C}^n$. By universality we mean that *any other* germ (∇',E',X',x'_o) of integrable deformation of ∇^o is induced by $(\nabla^{\rm un},\underline{\mathbb{C}^n},\mathbb{C}^n,\boldsymbol{u}_o)$ via a *unique* base change, i.e. via a unique germ of map $\varphi\colon(X',x'_o)\to(\mathbb{C}^n,\boldsymbol{u}_o)$. See Theorem 4.7.

In the case $A_o\in\mathcal{M}_{\rm reg}\cap\mathcal{M}_{\rm diag}$, a more detailed description of the universal integrable deformation $\nabla^{\rm un}$ is available. In this semisimple-regular case, it follows from independent results of B. Malgrange and M. Jimbo, T. Miwa, K. Ueno [JMU81, Mal83b, Mal86] that the germ of the universal integrable deformation $\nabla^{\rm un}$ can be defined (in a suitable basis of section of the trivial bundle $\underline{\mathbb{C}^n}$) by the matrix

$$\Omega_{\rm JMUM}(z,\boldsymbol{u})=-\left(-\Lambda(\boldsymbol{u})+\frac{1}{z}\left(\mathcal{B}'_o+[\Gamma(\boldsymbol{u}),\Lambda(\boldsymbol{u})]\right)\right){\rm d}z\\ -z{\rm d}\Lambda(\boldsymbol{u})-[\Gamma(\boldsymbol{u}),{\rm d}\Lambda(\boldsymbol{u})],\qquad \boldsymbol{u}\in\mathbb{D}\quad(1.7)$$

where

- $\mathbb{D}\subseteq\mathbb{C}^n$ is a sufficiently small polydisc centered at $\boldsymbol{u}_o\in\mathbb{C}^n$,
- $\Lambda(\boldsymbol{u})={\rm diag}(u^1,\dots,u^n)$, where $\boldsymbol{u}\in\mathbb{C}^n$,
- $\mathcal{B}'_o={\rm diag}(b_1,\dots,b_n)$ is a constant diagonal matrix,
- $\Gamma=(\Gamma_{ij})_{ij=1}^n$ is an off-diagonal matrix.

Moreover, these data are such that:

(1) there exists $P\in GL(n,\mathbb{C})$ such that $P^{-1}A_oP=\Lambda(\boldsymbol{u}_o)$, and the diagonal part of $P^{-1}B_oP$ equals $\mathcal{B}'_o$,
(2) and $\nabla^{\rm un}$ is formally equivalent, at $z=\infty$, to the matrix connection $-{\rm d}\,(z\Lambda(\boldsymbol{u}))-\mathcal{B}'_o\frac{{\rm d}z}{z}$.

As long as the matrix A_o is not a regular matrix, the classification problem of germs of integrable deformations of ∇^o becomes extremely more difficult[4]. In this paper, we address the classification problem of germs of integrable deformations of ∇^o, in the case $A_o\in\mathcal{M}_{\rm diag}$ only, i.e. A_o is a diagonalizable matrices with possibly non-simple spectrum.

[4]By quoting B. Malgrange [Mal83a, Rk. 3.8]: «[. . .] le problème de trouver les solutions [of the integrability equations] passant par un A_o non régulier semble très difficile.»

In the case $A_o \in \mathcal{M}_{\rm diag}$, in [Sab21] C. Sabbah considered a connection ∇^o satisfying the following *Property of Partial Non-Resonance* (for short, *Property PNR*):

Property PNR: There exists a matrix $P \in GL(n, \mathbb{C})$ diagonalizing A_o, i.e. $P^{-1}A_oP = \Lambda_o = \mathrm{diag}(u_o^1, \dots, u_o^n)$, and such that the matrix $\mathcal{B}_o := P^{-1}B_oP$ has the following properties:

($\star$) $\mathcal{B}_o'' \in \mathrm{Im}\,\mathrm{ad}(\Lambda(\boldsymbol{u}_o))$.

($\star\star$) $\mathcal{B}_o'$ is *partially non-resonant*, i.e. we have $(\mathcal{B}_o')_{ii} - (\mathcal{B}_o')_{jj} \notin \mathbb{Z} \setminus \{0\}$ whenever $u_o^i = u_o^j$.

Remarkably, under this assumption, Sabbah proved the existence of a germ of integrable deformation of ∇^o of the form (1.7), and satisfying the properties (1) and (2) above. For an analytical proof of Sabbah's result, see [Cot21a].

Although the integrable deformation constructed by Sabbah shares many properties of the Malgrange–Jimbo–Miwa–Ueno connection $\nabla^{\rm un}$, no claim of universality was formulated in [Sab21]. On the one hand, this is coherent with the fact that, in general, there is no versal deformation of ∇^o if A_o is not regular (see e.g. Appendix A.2). On the other hand, we will prove that Sabbah's connection still enjoys a *relative* universality property: this was one of the motivating themes of the current work.

Remark 1.2. When $n = 2$, the cases $A_o \in \mathcal{M}_{\rm reg}$ and $A_o \in \mathcal{M}_{\rm diag}$ are exhaustive. In such a case a complete classification of germs of integrable deformation has been developed in the very interesting preprint [Her21]. In the current paper, we address the case of arbitrary n, and we do not obtain a complete classification as in *loc. cit.*. Our results, however, still emphasize the richness of the classification in the non-regular case.

1.3. **Results.** The first main result of this paper is of preliminary nature, framed in the general study of operator-valued holomorphic functions. It consists of a characterization of matrix-valued holomorphic functions $A\colon X \to M(n, \mathbb{C})$, defined on a complex manifold X, which are *locally holomorphically Jordanizable*. We say that A is locally holomorphically Jordanizable at $x_o \in X$ if there exist an open neighborhood $U \subseteq X$ of x_o and a holomorphic function $P\colon U \to GL(n, \mathbb{C})$ such that $P^{-1}AP$ is in Jordan canonical form. Clearly, a necessary condition for A to be locally holomorphically Jordanizable at x_o is

(I) the existence of a holomorphic function $J\colon U \to M(n, \mathbb{C})$, in Jordan canonical form for any $x \in U$, and similar to $A(x)$ for any $x \in U$.

The validity of condition (I) only, however, is not sufficient. Set

- $\sigma(A(x))$ to be spectrum (i.e. the set of eigenvalues) of $A(x)$ for any $x \in X$;
- $\lambda_1, \dots, \lambda_r\colon U \to \mathbb{C}$ to be the holomorphic eigenvalues functions of $A|_U$ (without counting multiplicities);
- $\mathrm{coal}(A) \subseteq X$ to be the *coalescence locus* of A, namely the set of points $x \in X$ such that
$$\forall\, V \text{neigh. of } x,\ \exists\, z \in V : \mathrm{card}\,\sigma(A(z)) > \mathrm{card}\,\sigma(A(x));$$
- $\mathcal{G}_n$ to be the disjoint union of complex Grassmannians of subspaces in $\mathbb{C}^n$, i.e. $\mathcal{G}_n = \coprod_{k=0}^n G(k, n)$ (with the complex analytic topology).

In Theorem 3.21, we prove that A is locally holomorphically Jordanizable at x_o if and only if conditions (I) above, (II) and (III) below hold true:

(II) For each $i = 1, \dots, r$, the function $\psi_i\colon U \setminus \mathrm{coal}(A) \to \mathcal{G}_n$ defined by $x \mapsto \ker(A(x) - \lambda_i(x)\mathrm{Id}_n)^n$ admits a limit $L_i \in \mathcal{G}_n$ at $x = x_o$.

(III) We have $\bigoplus_{i=1}^r L_i = \mathbb{C}^n$.

Moreover, standing on results of W. Kaballo [Kab76], we prove that the limit L_i, with $i = 1, \dots, r$, necessarily equals the space $\mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x_o]$ of values at x_o of $\mathbb{C}^n$-valued holomorphic functions in the kernel sheaf $\mathscr{ker}(A - \lambda_i \mathrm{Id})^n$ of the morphism of $\mathscr{O}_X$-modules

$$(A - \lambda_i \mathrm{Id})^n\colon \mathscr{O}_X^{\oplus n} \to \mathscr{O}_X^{\oplus n}.$$

We also prove that condition (II) is equivalent to the condition

(II.bis) the function $U \ni x \mapsto \dim \mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x]$ is continuous at x_o, for any $i = 1, \dots, r$.

Theorem 3.21 extends to the case of several complex variables previous results of Ph.G.A. Thijsse and W. Wasow, independently obtained in [Thi85, Was85]. See Corollary 3.24.

Remark 1.3. Let us stress the main differences of our results with those of [Thi85, Was85]:

1. The results of both Thijsse and Wasow work under the assumption $\dim_{\mathbb{C}} X = 1$ (more precisely, $X \subseteq \mathbb{C}$ is an open region). In this case, condition (II) above is automatically satisfied. Thijsse realized this standing on the results of [BKL75]. Alternatively, we will deduce this from results of W. Kaballo [Kab76, Kab12].
2. Wasow's Theorem actually works under the further assumption $\mathrm{coal}(A) = \emptyset$. In such a case, also condition (III) is automatically satisfied.
3. Thijsse's and Wasow's results are statements of *global holomorphic similarity*. If $X \subseteq \mathbb{C}$ is an open region, and if A is locally holomorphically Jordanizable at each point $x \in X$, then one can prove that there exists a globally defined holomorphic matrix $T\colon X \to GL(n, \mathbb{C})$ such that $T(x)^{-1}A(x)T(x)$ is in Jordan form. This actually is an important peculiarity of all 1-dimensional Stein manifolds X, see [Gur88, Lei20, For17] and Theorem 3.1.

To the best of our knowledge, our characterization of locally holomorphically Jordanizable matrices, depending on several complex variables, was never explicitly formulated in literature. The local holomorphic Jordanizability condition plays a crucial condition in the theory of isomonodromic deformations at irregular singularities with coalescing eigenvalues [CG18, CDG19]. We expect that our characterization will be useful for generalizing results of [CG18, CDG19], as well as for the study of strata of Dubrovin–Frobenius and flat F-manifolds [CG17, CDG20, Cot21b].

The second main results of the paper concerns the classification problem of integrable deformations of ∇^o defined by (1.6). We first introduce several classes of germs of integrable deformations of ∇^o, namely:

- the class $\mathfrak{I}(\nabla^o)$ of *all* germs of integrable deformations of ∇^o;
- the class $\mathfrak{I}_{\mathrm{d}}(\nabla^o)$ of germs of *diagonal type* (d-*type*) integrable deformations of ∇^o, that is those integrable deformations whose pole and deformation parts (A and C in equation (1.4)) are locally holomorphically diagonalizable;
- the class $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$ of germs of *generic* d-type integrable deformations of ∇^o, that is those integrable deformations (∇, E, X, x_o) of ∇^o of d-type whose pole part has holomorphic diagonal form $\Delta_0 = \mathrm{diag}(f_1(x), \dots, f_n(x))$ with $\mathrm{d}_{x_o} f_i \neq \mathrm{d}_{x_o} f_j$ for $i \neq j$;

- the class $\mathfrak{I}_{\rm dv}(\nabla^o)$ of germs of *diagonal-vainshing type* (dv-*type*) integrable deformations of ∇^o, that is those integrable deformations (∇, E, X, x_o) which are defined (in a suitable basis) by a matrix of connection 1-forms
$$\widetilde{\Omega}(z,x) = -\left(\Delta_0(x) + \frac{1}{z}\mathcal{B}(x)\right)\mathrm{d}z - z\,\mathrm{d}\Delta_0(x) + \varpi(x),$$
where the Δ_0 and $\mathcal{B}$ are holomorphic matrix-valued functions, and ϖ is a holomorphic matrix-valued 1-form such that[5]
$$\Delta_0(x) = \mathrm{diag}(f_1(x), \dots, f_n(x)), \qquad \mathcal{B}'' = [\mathcal{L}, \Delta_0], \qquad \varpi'' = [\mathrm{d}\Delta_0, \mathcal{L}],$$
for a holomorphic matrix-value function $\mathcal{L}\colon X \to M(n, \mathbb{C})$, $\mathcal{L} = \mathcal{L}''$;
- the class $\mathfrak{I}_{\rm fs}(\nabla^o)$ of germs of *formally simplifiable* integrable deformations of ∇^o which are formally equivalent, at $z = \infty$, to the connection
$$\mathrm{d} - \mathrm{d}(z\Delta_0(x)) - \mathfrak{B}'_o\frac{\mathrm{d}z}{z},$$
where $\Delta_0(x) = \mathrm{diag}(f_1(x), \dots, f_n(x))$ and $\mathfrak{B}'_o$ is a diagonal constant matrix.

In Theorems 4.24 and 4.25 we show that

(1) if $A_o \in \mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$, then we have
$$\emptyset \neq \mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o) \subsetneq \mathfrak{I}_{\rm fs}(\nabla^o) = \mathfrak{I}_{\rm dv}(\nabla^o) = \mathfrak{I}_{\rm d}(\nabla^o) = \mathfrak{I}(\nabla^o),$$
(2) while if $A_o \in \mathcal{M}_{\rm diag}$ and the Property PNR holds true, then we have a more rich classification, since
$$\emptyset \neq \mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o) \subsetneq \mathfrak{I}_{\rm fs}(\nabla^o) \subseteq \mathfrak{I}_{\rm dv}(\nabla^o) \subseteq \mathfrak{I}_{\rm d}(\nabla^o) \subseteq \mathfrak{I}(\nabla^o),$$
where all the inclusions are in general strict (see e.g. Example 4.21).

For clarity of exposition, let us denote with the unified notation[6] $\nabla^{\rm JMUMS}$ the integrable deformation of ∇^o constructed by Jimbo–Miwa–Ueno–Malgrange (in the case $A_o \in \mathcal{M}_{\rm diag} \cap \mathcal{M}_{\rm reg}$), and by Sabbah (in the case $A_o \in \mathcal{M}_{\rm diag}$ & Property PNR). In Theorem 4.31 we further show the existence and uniqueness of a class $\mathfrak{I}_{\rm JMUMS}$ of germs of integrable deformations of ∇^o such that

(1) the integrable deformation $\nabla^{\rm JMUMS}$ is an element of $\mathfrak{I}_{\rm JMUMS}$,
(2) any germ of integrable deformation, which is induced by $\nabla^{\rm JMUMS}$ via a base change map, is an element of $\mathfrak{I}_{\rm JMUMS}$,
(3) $\mathfrak{I}_{\rm JMUMS}$ is maximal with respect to the properties above.

Moreover, we show that if a germ of integrable deformation is induced by $\nabla^{\rm JMUMS}$ via a base change map, then the germ of such a map is unique. For short, we say that $\nabla^{\rm JMUMS}$ is $\mathfrak{I}_{\rm JMUMS}$-*universal*, see Definition 4.29.

In the case $A_o \in \mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$, we necessarily have $\mathfrak{I}_{\rm JMUMS} = \mathfrak{I}(\nabla^o)$ by universality of the JMUM integrable deformation. In the more general case $A_o \in \mathcal{M}_{\rm diag}$ & Property PNR, we show that $\mathfrak{I}_{\rm JMUMS}$ is large enough to satisfy the inequality
$$\mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o) \subseteq \mathfrak{I}_{\rm JMUMS} \subseteq \mathfrak{I}_{\rm fs}(\nabla^o). \tag{1.8}$$

[5]Here, and in the main body of the text, we denote by M' the diagonal part of a matrix M, and by $M'' = M - M'$ its off-diagonal part.

[6]Here JMUMS stands for Jimbo–Miwa–Ueno–Malgrange–Sabbah.

In addition to that, in Appendix A.3 we show that the validity of the PNR condition is a sharp condition for the results above. More precisely, we exhibit a connection ∇^o with $A_o \in \mathcal{M}_{\mathrm{diag}}$ and for which the Property PNR cannot hold true: we show that the germs in $\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)$ cannot be induced by a single germ of integrable deformation, in opposition to (1.8).

The proof of the left inequality in (1.8) is based on a remarkable initial value property of an overdetermined system of non-linear PDEs, that we call *generalized Darboux–Egoroff equations*. Consider n holomorphic functions $f_1(\boldsymbol{x}), \dots, f_n(\boldsymbol{x})$ in d complex variables $\boldsymbol{x} = (x^1, \dots, x^d)$. Let $b_1, \dots, b_n \in \mathbb{C}$ be arbitrary constants. The generalized Darboux–Egoroff system $\mathrm{DE}_{d,n}((f_i)_{i=1}^n; (b_i)_{i=1}^n)$, in the $n^2 - n$ unknown functions $(F_{kh}(\boldsymbol{x}))_{k,h=1}^n$, with $k \neq h$, is given by

$$
\begin{aligned}
&(\partial_j f_h - \partial_j f_k)\partial_i F_{kh} - (\partial_i f_h - \partial_i f_k)\partial_j F_{kh} = \\
&\qquad\qquad \sum_{\ell=1}^n (\partial_i f_\ell - \partial_i f_k)(\partial_j f_h - \partial_j f_\ell) F_{k\ell}F_{\ell h} - \sum_{\ell=1}^n (\partial_j f_\ell - \partial_j f_k)(\partial_i f_h - \partial_i f_\ell) F_{k\ell}F_{\ell h},
\end{aligned}
$$

$$
\begin{aligned}
&(f_h - f_k)\partial_i F_{kh} = (b_h - b_k - 1)(\partial_i f_h - \partial_i f_k) F_{kh} \\
&\qquad\qquad + \sum_{\ell=1}^n (\partial_i f_\ell - \partial_i f_k)(f_h - f_\ell) F_{k\ell}F_{\ell h} - \sum_{\ell=1}^n (f_\ell - f_k)(\partial_i f_h - \partial_i f_\ell) F_{k\ell}F_{\ell h},
\end{aligned}
$$

for any $i, j = 1, \dots, d$, and any $k, h = 1, \dots, n$, with $k \neq h$. The solutions F_{kh} of this system can be arranged in an off-diagonal matrix $F \colon U \subseteq \mathbb{C}^d \to M(n, \mathbb{C})$.

Theorem 4.28, which is the third main result of the paper, asserts that

- if $\boldsymbol{x}_o \in \mathbb{C}^d$,
- if $\mathrm{d}_{\boldsymbol{x}_o} f_k \neq \mathrm{d}_{\boldsymbol{x}_o} f_k$ for any $h, k = 1, \dots, n$, with $k \neq h$,
- if $b_h - b_k \notin \mathbb{Z} \setminus \{0\}$ whenever $f_h(\boldsymbol{x}_o) = f_k(\boldsymbol{x}_o)$,

then any formal power series solution F of $\mathrm{DE}_{d,n}((f_i)_{i=1}^n; (b_i)_{i=1}^n)$, centered at $\boldsymbol{x}_o$, is uniquely determined by its initial value $F(\boldsymbol{x}_o)$. Consequently, the same result holds for any analytic solution defined in a neighborhood of $\boldsymbol{x}_o$. This generalizes the results of [Cot21a, Lemma 5.21] [Cot21b, Lemma 6.16] for the (standard) Darboux–Egoroff equations.

1.4 **Structure of the paper.** In Section 2, we review the theory of bundles of matrices, introduced by V.I. Arnol'd. Special emphasis is given on both geometrical and combinatorial aspects of the theory. In particular, we review its connection with the theory of double partitions of integers (an aspect which is missing in the original source [Arn71]), as well as some of the main results on the semi-algebraic stratification of the space of matrices defined by the bundles.

Section 3 is devoted to the study of matrix-valued holomorphic maps which are locally/globally holomorphically similar. After reviewing known results due to R.M. Guralnick, J. Leiterer, and F. Forstnerič, we address the study of Jordan forms of matrix-valued holomorphic maps. The first main result of the paper, Theorem 3.21, is formulated and proved.

In Section 4, we first review known results on families of RHB problems due to B. Malgrange, M. Jimbo, T. Miwa, K. Ueno, and C. Sabbah. We subsequently introduce several classes (d/dv/fs-types) of germs of integrable deformations of a solution of a RHB problem, and we study their inclusive relations, see Theorems 4.24 and 4.25. We introduce the notion of $\mathfrak{I}$-universality, and we prove that the Sabbah's integrable deformation satisfies a relative universal property for a suitable maximal class $\mathfrak{I}_{\rm JMUMS}$, see Theorem 4.31. Furthermore, we study the generalized Darboux–Egoroff system of PDEs, and we prove its initial value property, Theorem 4.28.

In Appendix A.1, we recall Malgrange's proof of existence of a universal integrable deformation for the connection (1.6) with $A_o \in \mathcal{M}_{\rm reg}$.

In Appendix A.2, we show via an example that if $A_o \notin \mathcal{M}_{\rm reg}$, then in general there is no versal integrable deformation of (1.6).

In Appendix A.3, we show via an example that the Property PNR is a sharp condition for the existence of an integrable deformation inducing all germs of integrable deformations of generic d-type.

Acknowledgements. The author is thankful to R. Conti, G. Degano, D. Guzzetti, C. Hertling, P. Lorenzoni, D. Masoero, A.T. Ricolfi, and C. Sabbah for several valuable discussions. This research was supported by the FCT Project PTDC/MAT-PUR/ 30234/2017 "Irregular connections on algebraic curves and Quantum Field Theory".

2. Bundles of matrices

In what follows by *multiset* we mean a pair (A, m) where A is a finite set and $m\colon A \to \mathbb{N}^*$ is an arbitrary function, called *multiplicity function*. More informally, a multiset consists of a finite collection of objects (the elements) which may occur more than once: the element $a \in A$ will occur exactly $m(a)$ times.

For short, we will represent multisets by listing their elements, with multiplicity, between $\{\!\{\dots\}\!\}$ brackets.

Partitions of integers provide examples for multisets. Given a non-negative number n, a multiset $\lambda = \{\!\{\lambda_1, \dots, \lambda_r\}\!\}$ of non-negative integers is a partition of n provided that $n = \sum_{i=1}^{r} \lambda_i$. We denote by $\mathscr{P}(n)$ the set of partitions of n.

2.1. Double partitions.

A *double partition* of a positive integer n is a double sum representation of n as follows

$$n = n_1 + \dots + n_k, \qquad n_j = b_{j1} + \dots + b_{jl_j}, \quad j = 1, \dots, k, \tag{2.1}$$

with $b_{ij} \in \mathbb{Z}_{>0}$, and where $k, l_1, \dots, l_k$ are arbitrary positive integers. The numbers $n_1, \dots, n_k$ are the *rough parts* of the double partition, whereas the numbers b_{ij} are its *fine parts*. The order of rough parts, and of fine parts –inside a single rough part– is not relevant.

A double partition of n can thus be identified with the datum of a multiset of ordinary partitions of the summands n_i's, with $i = 1, \dots, k$. We denote by $\mathscr{P}(2, n)$ the set of double partitions of n, and by $p(2, n)$ its cardinality.

For short, we use the notation $\boldsymbol{b} = \{\!\{b_{11}, \dots, b_{1l_1}; \dots; b_{k1}, \dots, b_{kl_k}\}\!\}$ for the multiset of partitions. The integer k is the *rough length* of $\boldsymbol{b}$, denoted by $\|\boldsymbol{b}\|$, whereas the integers $l_1, \dots, l_k$ are the *fine lengths* of $\boldsymbol{b}$.

If $\boldsymbol{b} = \{\!\{b_{11}, \dots, b_{1l_1}; \dots; b_{k1}, \dots, b_{kl_k}\}\!\} \in \mathscr{P}(2, n)$, without loss of generality we may (and will) assume that $\boldsymbol{b}$ is *monotonically ordered*: this means that $b_{j1} \geqslant b_{j2} \geqslant \dots \geqslant b_{jl_j}$ for any $j = 1, \dots, k$.

2.2. **Bundles of matrices.** Given $n \in \mathbb{N}^*$, we denote by $M(n, \mathbb{C})$ the complex vector space of $n \times n$-matrices.

Two matrices $A_1, A_2 \in M(n, \mathbb{C})$ have the *same Jordan type* if their Jordan forms differ by their eigenvalues only, the number of distinct eigenvalues and the orders of the Jordan blocks being the same. A *bundle of matrices* is a maximal set of matrices with the same Jordan type.

Bundles of matrices are in 1-1 correspondence with double partitions of n.

Given the double partition $\boldsymbol{\lambda} = \{\!\{\lambda_{11}, \dots, \lambda_{1l_1}; \dots; \lambda_{k1}, \dots, \lambda_{kl_k}\}\!\}$, we define $\mathcal{M}_{\boldsymbol{\lambda}}$ to be set the of matrices with Jordan form

$$\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l_i} J_{\lambda_{ij}}(\mu_i), \qquad J_h(\mu) = \begin{pmatrix} \mu & 1 & 0 & \dots \\ 0 & \mu & 1 & \dots \\ \vdots & & \ddots & \\ 0 & \dots & & \mu \end{pmatrix} \in M(h, \mathbb{C}). \tag{2.2}$$

The partition $\{\!\{\lambda_{i1}, \dots, \lambda_{il_i}\}\!\}$ is called the *Segre characteristic* of the eigenvalue μ_i.

The space $M(n, \mathbb{C})$ admits thus a decomposition into bundles,

$$M(n, \mathbb{C}) = \coprod_{\boldsymbol{\lambda} \in \mathscr{P}(2,n)} \mathcal{M}_{\boldsymbol{\lambda}},$$

where each bundle $\mathcal{M}_{\boldsymbol{\lambda}}$ is a smooth semi-algebraic submanifold of $M(n, \mathbb{C})$.

We will also adopt the following notation: the bundle $\mathcal{M}_{\boldsymbol{\lambda}}$, associated with $\boldsymbol{\lambda} \in \mathscr{P}(2, n)$ will be labelled by the string

$$\mu_1^{\lambda_{11}} \mu_1^{\lambda_{12}} \dots \mu_2^{\lambda_{21}} \mu_2^{\lambda_{22}} \dots \mu_k^{\lambda_{k1}} \mu_k^{\lambda_{k2}} \dots$$

where $\mu_1, \dots, \mu_k$ denote distinct eigenvalues.

For example, for $n = 2$, we have 3 bundles of matrices, labelled by

$$\mu_1\mu_2, \quad \mu_1^2, \quad \mu_1\mu_1,$$

which correspond to the Jordan forms

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \qquad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}, \qquad \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \end{pmatrix}$$

respectively.

For $n = 3$, we have 6 bundles of matrices, labelled by

$$\mu_1\mu_2\mu_3, \quad \mu_1^2\mu_2, \quad \mu_1\mu_1\mu_2, \quad \mu_1^3, \quad \mu_1^2\mu_1, \quad \mu_1\mu_1\mu_1.$$

2.3. **Number of bundles.** By the preceding paragraph, the number of bundles of matrices in $M(n, \mathbb{C})$ equals the number $p(2, n)$ of double partitions of n. The first values of $p(2, n)$, for $n = 1, \dots, 20$, are

$$1,\ 3,\ 6,\ 14,\ 27,\ 58,\ 111,\ 223,\ 424,\ 817,\ 1527,\ 2870,\ 5279,\ 9710,\ 17622,\\ 31877,\ 57100,\ 101887,\ 180406,\ 318106, \dots$$

The history of this numerical sequence is quite rich and interesting. In 1854, A. Cayley first introduced and studied the sequence of numbers $p(2, n)$, see [Cay55]. He claimed that the same numbers arise in the classification of homographies of the projective space $\mathbb{P}^n$. Furthermore, Cayley noticed that the first numbers of this sequence (minus 2, and with some computational mistakes for $n = 5, 7, 8$) appeared in [Syl51], a study of intersections of quadrics by J.J. Sylvester. Subsequently, all these research directions were extensively developed by C. Segre. In 1883, C. Segre completed the classification of intersection of two quadrics in a projective space, one of the main topics of his *Tesi di Laurea* [Seg83, Pt. II, §3]. Moreover, in [Seg12, Ch. II.14] Segre presented a complete classification of collineations in projective spaces. In both classification problems, a multiset of partitions (the *Segre symbol*) is a classifying invariant. Segre symbols are indeed double partitions of integers, and their number thus equal $p(2, n)$. For further details see [HP94a, Book II, Ch. VIII] [HP94b, Book IV, Ch. XIII, §10–11]. See also [Bel16, FMS21].

Introduce the generating function

$$\mathcal{P}(2; z) := \sum_{n=0}^{\infty} p(2, n) z^n, \quad p(2, 0) := 1.$$

Denote by $p(n)$ the number of ordinary partitions of n.

Theorem 2.1 ([Cay55]). *We have*

$$\mathcal{P}(2; z) = \prod_{n=1}^{\infty} (1 - z^n)^{-p(n)}.$$

Proof. For $n \in \mathbb{N}$, the number $p(2, n)$ equals the number of representations of n as following sums

$$n = \sum_{k=1}^{\infty} k(m_{k,1} + \dots + m_{k,p(k)}), \quad m_{i,j} \geqslant 0.$$

This follows from the interpretation of a double partition as a multiset of partitions, and the $m_{i,j}$'s represent the multiplicities. We have

$$\prod_{n=1}^{\infty} (1 - z^n)^{-p(n)} = \prod_{n=1}^{\infty} \sum_{\substack{m_j=0 \\ 1 \leqslant j \leqslant p(n)}}^{\infty} z^{n(m_1 + \dots + m_{p(n)})} = \sum_{\substack{m_{i,j}=0 \\ 1 \leqslant i,j}}^{\infty} z^{1m_{1,1} + 2(m_{2,1} + m_{2,2}) + \dots}. \qquad \square$$

Corollary 2.2. *We have the recursive formula*

$$p(2, n) = \frac{1}{n} \sum_{k=1}^{n} \sigma(k) p(2, n-k), \quad \sigma(k) := \sum_{d|k} d \cdot p(d).$$

Proof. From Theorem 2.1, we obtain $\log \mathcal{P}(2,z) = \sum_{n,m=1}^{\infty} \frac{p(n)}{m} z^{nm}$. The recursive formula follows from the identity $\frac{d}{dz}\mathcal{P}(2,z) = \mathcal{P}(2,z)\frac{d}{dz}(\log \mathcal{P}(2,z))$. □

The numbers $p(2,n)$ have an exponential growth, as described by the following result, due to R. Kaneiwa and V. M. Petrogradsky, which provides an analog of the well-known Hardy–Ramanujan asymptotic formula for $p(n)$, see [HR18].

Theorem 2.3 ([Kan79, Kan80, Pet99]). *We have the following asymptotic expansion*

$$\ln p(2,n) = \left(\frac{\pi^2}{6} + o(1)\right)\frac{n}{\ln n}, \quad n \to \infty. \qquad \square$$

Remark 2.4. Although [Kan79, Kan80] provide more precise terms of the expansion, the results of [Pet99] have wider implications. Ordinary partitions and double partitions are just the first instances of *r-fold partitions*, defined as representations of an integer $n \in \mathbb{N}$ as r-fold sums of non-negative integers, see [Kan79]. Denoting by $p(r,n)$ the number of r-fold partitions of n, with $p(r,0) := 1$ for any $r \geqslant 1$, the generating function $\mathcal{P}(r,z) := \sum_n p(r,n)z^n$ satisfies the identity

$$\mathcal{P}(r,z) = \prod_{n=1}^{\infty}(1-z^n)^{-p(r-1,n)}, \quad r \geqslant 2.$$

See [Kan79]. As an application of [Pet99, Th. 2.1], one obtains that

$$\ln p(r,n) = \left(\frac{\pi^2}{6} + o(1)\right)\frac{n}{\ln^{(r-1)} n}, \quad n \to \infty,$$

where $\ln^{(k)} x := \underbrace{\ln\ln\dots\ln}_{k \text{ times}} x$ for $k \geqslant 1$. More general applications of [Pet99, Th. 2.1] allows an estimate of the growth of the number of some generalized partitions, and growth of free polynilpotent finitely generated Lie algebras. See also [Pet00].

2.4. **Bundles as fibered spaces.** Given $\boldsymbol{a} = (a_1,\dots,a_h) \in \mathbb{N}^h$, with $h \geqslant 1$, set $|\boldsymbol{a}| := \sum_{j=1}^{h} a_j$. Let $\mathfrak{S}_{\boldsymbol{a}}$ to be the subgroup of the symmetric group $\mathfrak{S}_{|\boldsymbol{a}|}$ defined by $\mathfrak{S}_{\boldsymbol{a}} := \mathfrak{S}_{a_1} \times \dots \times \mathfrak{S}_{a_h}$, where $\mathfrak{S}_{a_i}$ is the symmetric group on the elements $\{(\sum_{k=1}^{i-1} a_i) + 1, (\sum_{k=1}^{i-1} a_i) + 2, \dots, \sum_{k=1}^{i} a_i\}$, for $i = 1,\dots,h$.

Define the *configuration space $\mathcal{C}_{\boldsymbol{a}}$ of $|\boldsymbol{a}|$ colored points in the plane* as the quotient

$$\mathcal{C}_{\boldsymbol{a}} := \left(\mathbb{C}^{|\boldsymbol{a}|} \setminus \Delta\right)/\mathfrak{S}_{\boldsymbol{a}},$$

where Δ be the union of big diagonal hyperplanes in $\mathbb{C}^n$, defined by the equations

$$\Delta := \bigcup_{i<j}\{\boldsymbol{u} \in \mathbb{C}^n : u^i = u^j\}.$$

The tuple $\boldsymbol{a}$ dictates the coloring of the points: in total we have h colors, and for each $i = 1,\dots,h$ we have a_i points with the same i-th color. The order of points with the same color is not relevant.

The bundles $\mathcal{M}_{\boldsymbol{\lambda}}$ are fibered spaces over suitable colored configuration spaces $\mathcal{C}_{\boldsymbol{a}}$. More precisely, for each $\boldsymbol{\lambda} \in \mathscr{P}(2,n)$ denote by $\boldsymbol{m}_{\boldsymbol{\lambda}} = (m_1,\dots,m_h) \in \mathbb{N}^h$ the tuple of multiplicities

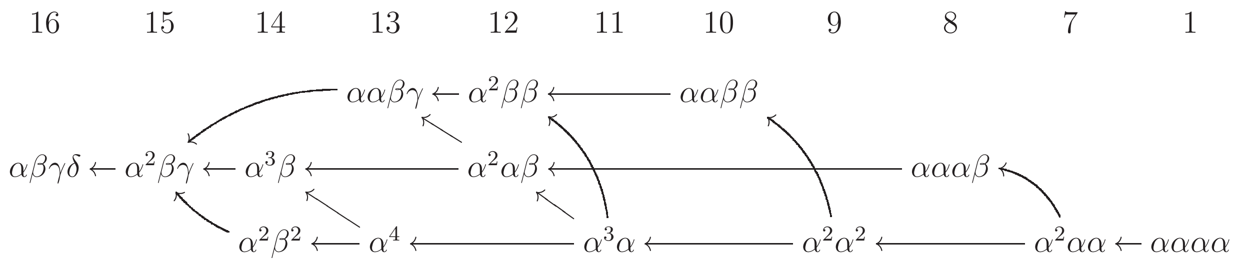

FIGURE 1. Hasse diagram of the closure stratification of bundles in $M(4,\mathbb{C})$. The numbers on the top row denote the dimensions of the bundles.

of the elements of the multiset $\boldsymbol{\lambda}$. We have a natural map $\pi_{\boldsymbol{\lambda}}\colon \mathcal{M}_{\boldsymbol{\lambda}} \to \mathcal{C}_{\boldsymbol{m}_{\boldsymbol{\lambda}}}$, defined by associating to a matrix A its *colored spectrum*, i.e. the set of its *colored distinct eigenvalues*. Two distinct eigenvalues have the same color if they have the same Segre characteristics. The fibers of $\pi_{\boldsymbol{\lambda}}$ are similarity orbits of the Jordan matrices (2.2).

Theorem 2.5 ([Arn71])**.** *The codimension of the bundle $\mathcal{M}_{\boldsymbol{\lambda}}$ equals*

$$c = \left[\sum_{j=1}^{\|\boldsymbol{\lambda}\|} \lambda_{j1} + 3\lambda_{j2} + 5\lambda_{j3} + \dots\right] - \|\boldsymbol{\lambda}\|.$$

Proof. The dimension of $\mathcal{M}_{\boldsymbol{\lambda}}$ equals the dimension of the base space $\mathcal{C}_{\boldsymbol{m}_{\boldsymbol{\lambda}}}$ plus the dimension of the fiber. We have $\dim \mathcal{C}_{\boldsymbol{m}_{\boldsymbol{\lambda}}} = \|\boldsymbol{\lambda}\|$, and the codimension of the similarity orbit of the Jordan matrix (2.2) equals the first summand above. See [Gan59, Ch. VIII]. $\square$

2.5. **Stratification of bundles.** The decomposition $M(n,\mathbb{C}) = \coprod_{\boldsymbol{\lambda}\in\mathscr{P}(2,n)} \mathcal{M}_{\boldsymbol{\lambda}}$ defines a semi-algebraic *stratification* of the space of matrices: the family $(\mathcal{M}_{\boldsymbol{\lambda}})_{\boldsymbol{\lambda}\in\mathscr{P}(2,n)}$ is locally finite, and if $\mathcal{M}_{\boldsymbol{\lambda}} \cap \overline{\mathcal{M}_{\boldsymbol{\nu}}} \neq \emptyset$ then $\mathcal{M}_{\boldsymbol{\lambda}} \subseteq \overline{\mathcal{M}_{\boldsymbol{\nu}}}$.

Define the relation $\trianglelefteq$ on the set of bundles in $M(n,\mathbb{C})$ by

$$\mathcal{M}_{\boldsymbol{\lambda}} \trianglelefteq \mathcal{M}_{\boldsymbol{\nu}} \quad :\Longleftrightarrow \quad \mathcal{M}_{\boldsymbol{\lambda}} \subseteq \overline{\mathcal{M}_{\boldsymbol{\nu}}}.$$

The relation $\trianglelefteq$ defines a partial ordering on bundles, called *closure relation*. Such a relation has been extensively studied in [BHM98, DE95, EEK97, EEK99, EJK03]. A convenient way to visualize[7] the closure stratification of bundles is via the Hasse diagram of the relation $\trianglelefteq$: each bundle $\mathcal{M}_{\boldsymbol{\lambda}}$ is represented by a point in the plane (the vertices of the diagram), and one draws an arrow from v_1 to v_2 if v_2 *covers* v_1 (i.e. we have $v_1 \trianglelefteq v_2$ and there is no v_3 such that $v_1 \trianglelefteq v_3 \trianglelefteq v_2$). See Figure 1.

Introduce the following two types of elementary transformations of $\boldsymbol{\lambda}$:

Type I. $\boldsymbol{\lambda} \to \boldsymbol{\lambda}'$, where $\boldsymbol{\lambda}'$ is obtained by "merging" two distinct partitions inside $\boldsymbol{\lambda}$, say $\{\!\{\lambda_{i1},\dots,\lambda_{il_i}\}\!\}$ and $\{\!\{\lambda_{j1},\dots,\lambda_{jl_j}\}\!\}$ with $i \neq j$, in the single one

$$\lambda_{i1} + \lambda_{j1} \geqslant \lambda_{i2} + \lambda_{j2} \geqslant \lambda_{i3} + \lambda_{j3} \geqslant \dots$$

[7] We invite the reader to use the software StratiGraph, developed at the Umeå University (Sweden), to visualize the Hasse diagrams in low dimensions. The software is available at the web-page https://www.umu.se/en/research/projects/stratigraph-and-mcs-toolbox/

The resulting rough length $\|\boldsymbol{\lambda}'\|$ equals $\|\boldsymbol{\lambda}\|-1$.

Type I. $\boldsymbol{\lambda}\to\boldsymbol{\lambda}'$, where $\boldsymbol{\lambda}'$ is obtained as follows. Fix $i\in\{1,\dots,k\}$, and consider the Ferrers diagram corresponding[8] to the partition $\lambda_i=\{\!\{\lambda_{i1},\dots,\lambda_{il_i}\}\!\}$ inside $\boldsymbol{\lambda}$. Then move one box rightward one column, or downward one raw, so long as the corresponding partitions λ_i' remains monotonic. The double partition $\boldsymbol{\lambda}'$ is obtained by $\boldsymbol{\lambda}$ by replacing $\lambda_i\mapsto\lambda_i'$. The resulting rough length $\|\boldsymbol{\lambda}'\|$ equals $\|\boldsymbol{\lambda}\|$.

Theorem 2.6 ([EEK99, Th. 2.6])**.** *Let*

$$\boldsymbol{\lambda}=\{\!\{\lambda_{11},\dots,\lambda_{1l_1};\dots;\lambda_{k1},\dots,\lambda_{kl_k}\}\!\},\qquad \boldsymbol{\lambda}'=\{\!\{\lambda'_{11},\dots,\lambda'_{1l_1'};\dots;\lambda'_{k'1},\dots,\lambda'_{k'l_k'}\}\!\},$$

be two double partitions of n, with $k'\leqslant k$. We have $\mathcal{M}_{\boldsymbol{\lambda}'}\trianglelefteq\mathcal{M}_{\boldsymbol{\lambda}}$ if and only if $\boldsymbol{\lambda}'$ can be obtained from $\boldsymbol{\lambda}$ via a finite sequence of transformations of type I and II.

If $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are related by a single transformation of type I and II, then there is an arrow from $\boldsymbol{\lambda}'$ to $\boldsymbol{\lambda}$ in the Hasse diagram of the closure relation $\trianglelefteq$. ☐

Remark 2.7. Transformations of Type I correspond to coalescences/splittings of eigenvalues.

The following result gives some insights into the difficulty of the decision procedure for testing the closure relation.

Theorem 2.8 ([EEK99, Th. 2.7])**.** *Deciding whether a bundle is in the closure of another bundle is an NP-complete problem.* ☐

2.6. **Conjugate bundles, and sets $\mathcal{M}_{\rm reg}$, $\mathcal{M}_{\rm diag}$.** The set $\mathscr{P}(n)$ of ordinary partitions of n is equipped with a natural involution $(-)^\vee\colon\mathscr{P}(n)\to\mathscr{P}(n)$, called *conjugation.*

The easiest way to define it is in terms of Ferrers diagrams. Given $\lambda\in\mathscr{P}(n)$, with associated Ferrers diagram F_λ, the conjugate partition $\lambda^\vee$ is the one associated with the transposed Ferrers diagram F_λ^T (i.e. the diagram obtained by flipping F_λ along its main diagonal, by turning rows to columns, and vice-versa).

This involution $\lambda\mapsto\lambda^\vee$ naturally extends to double partitions. Given a double partition $\boldsymbol{\lambda}=\{\!\{\lambda_1,\dots,\lambda_r\}\!\}$ in $\mathscr{P}(2,n)$, its conjugate $\boldsymbol{\lambda}^\vee$ is obtained by applying $(-)^\vee$ elementwise, that is

$$\boldsymbol{\lambda}^\vee=\{\!\{\lambda_1^\vee,\dots,\lambda_r^\vee\}\!\}.$$

We say that two bundles $\mathcal{M}_{\boldsymbol{\lambda}_1}$ and $\mathcal{M}_{\boldsymbol{\lambda}_2}$ are conjugate, if $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ are conjugate double partitions via $(-)^\vee$.

Introduce the following two subsets $\mathcal{R},\mathcal{D}\subseteq\mathscr{P}(2,n)$: let $\mathcal{R}$ to be the set of double partitions of n whose fine lengths equal 1, and let $\mathcal{D}$ to be the set of double partitions whose fine parts equal 1.

Lemma 2.9. *Elements of $\mathcal{R}$ are conjugate of elements of $\mathcal{D}$, and vice-versa.* ☐

Define the following unions of bundles

$$\mathcal{M}_{\rm reg}:=\coprod_{\boldsymbol{\lambda}\in\mathcal{R}}\mathcal{M}_{\boldsymbol{\lambda}},\qquad \mathcal{M}_{\rm diag}:=\coprod_{\boldsymbol{\lambda}\in\mathcal{D}}\mathcal{M}_{\boldsymbol{\lambda}}.$$

[8]In the first column we have λ_{i1} boxes, in the second column we have λ_{i2} boxes, ...and so on.

The set $\mathcal{M}_{\rm reg}$ equals the set of *regular matrices*, that is the set of matrices $A \in M(n,\mathbb{C})$ satisfying one (and hence all) of the following equivalent conditions:

1. the characteristic polynomial of A equals its minimal polynomial,
2. the centralizer of A in $M(n,\mathbb{C})$ is of minimal dimension (i.e. it equals n),
3. the centralizer of A in $M(n,\mathbb{C})$ is $\mathbb{C}[A]$.

The set $\mathcal{M}_{\rm diag}$ is the set of diagonalizable matrices, with possibly non-simple spectrum.

3. On the similarity and the Jordan forms of holomorphic matrices

3.1. Global and local holomorphic similarity. Let X be a complex manifold. Two holomorphic maps $A_1, A_2\colon X \to M(n,\mathbb{C})$ are said to be

- (*globally*) *holomorphically similar* on X if there exists a holomorphic map $H\colon X \to GL(n,\mathbb{C})$ such that $A_1 = H^{-1}A_2H$.
- *locally holomorphically similar at* $x_o \in X$ if there exists a neighborhood $U \subseteq X$ of x_o such that $A_1|_U$ and $A_2|_U$ are holomorphically similar on U.

Analogue definitions of *continuous* or $\mathcal{C}^k$*-smooth similarity*, with $0 \leqslant k \leqslant \infty$, can be given, according to the regularity of the matrix-valued function H above.

The problem of upgrading local to global holomorphic similarity has been addressed in [Gur88, Lei20, For17]. The following positive results have been obtained, provided that X is a Stein space.

Theorem 3.1 ([Gur88, Lei20, For17])**.** *Let X be a one dimensional Stein manifold, and $A_1, A_2\colon X \to M(n,\mathbb{C})$ be two holomorphic maps. If A_1, A_2 are locally holomorphically similar at each point of X, then they are globally holomorphically similar on X.* □

In the original proof of R.M. Guralnick, X is assumed to be a noncompact connected Riemann surface only. Recently, J. Leiterer extended Guralnick's result to all one dimensional Stein spaces (not necessarily smooth): his proof is based on the Oka principle for Oka pairs by O. Forster and K.J. Ramspott [FR66]. An alternative proof of this result also appears in the book [For17], where it is invoked an alternative Oka principle established in [For03].

For Stein spaces of arbitrary dimensions, we have the following result, which requires stronger assumptions.

Theorem 3.2 ([Lei20, Th. 1.4])**.** *Let X be a Stein space, and $A_1, A_2\colon X \to M(n,\mathbb{C})$ be two holomorphic maps such that:*

1. *A_1 and A_2 are globally continuously similar on X, i.e. there exists a continuous map $C\colon X \to M(n,\mathbb{C})$ such that $A_1 = C^{-1}A_2C$,*
2. *A_1 and A_2 are locally holomorphically similar at each point of X, i.e. for each $x_o \in X$ there is a neighborhood U_o of x_o, and a holomorphic map $H_o\colon U_o \to GL(n,\mathbb{C})$ with $A_1 = H_o^{-1}A_2H_o$ on U_o,*
3. *we have $H_o(x_o) = C(x_o)$ for each $x_o \in X$.*

Then A_1 and A_2 are globally holomorphically similar on X. □

Conditions (1) and (2) alone do not imply global holomorphic similarity. For a counterexample, see [Lei20, Th. 8.2].

3.2. **Three criteria for local holomorphic similarity.** Let X be a complex manifold, $A_1, A_2 \colon X \to M(n,\mathbb{C})$ be two holomorphic maps, $x_o \in X$, and $\Phi \in M(n,\mathbb{C})$ such that $\Phi A_1(x_o) = A_2(x_o)\Phi$.

Below are some criteria on X, A_1, A_2, Φ which will allow to infer local holomorphic similarity of A_1 and A_2 on a neighborhood of x_o. Following Leiterer, we will call them *Wasow's*, *Smith's*, and *Spallek's criterion* respectively.

Wasow's criterion: The dimension of the complex vector space

$$\{\Theta \in M(n,\mathbb{C}) \colon \Theta A_1(x) = A_2(x)\Theta\}$$

is constant for x in some neighborhood of x_o.

Smith's criterion: The space X is one dimensional, and there exist a neighborhood V_o of x_o and a continuous map $C_o \colon V_o \to M(n,\mathbb{C})$ such that $C_o A_1 = A_2 C_o$ on V_o, and $C_o(x_o) = \Phi$.

Spallek's criterion: There exist a neighborhood V_o of x_o and a smooth map $T_o \colon V_o \to M(n,\mathbb{C})$ such that $T_o A_1 = A_2 T_o$ on V_o, and $T_o(x_o) = \Phi$.

Theorem 3.3 ([Lei20, Th. 1.5])**.** *If one of the criteria above holds, then there exists a neighborhood U_o of x_o and a holomorphic map $H_o \colon U_o \to M(n,\mathbb{C})$ such that $H_o A_1 = A_2 H_o$ on U_o, and $H_o(x_o) = \Phi$. In particular, if Φ is invertible, then A_1 and A_2 are locally holomorphically similar.* □

Remark 3.4. The names of the criteria are justified as follows. In [Was62], W. Wasow formulated the first criterion and proved the statement of Theorem 3.3 under the unnecessary assumption that X is a domain in $\mathbb{C}$. The proof of Theorem 3.3 under the Smith's criterion is based on applications of the Smith factorization theorem, see [Jac75, Ch. III, Sec. 8] [GL09, Th. 4.3.1]. On the other hand, the proof of Theorem 3.3 under the Spallek's criterion follows from a special case of a result of K. Spallek, see [Spa65, Satz 5.4] [Spa67, Introduction].

Spallek's criterion and Theorem 3.2 imply the following result.

Corollary 3.5. *Let X be a Stein manifold. Let $A_1, A_2 \colon X \to M(n,\mathbb{C})$ be two holomorphic maps, globally $\mathcal{C}^\infty$-smoothly similar on X. Then A_1 and A_2 are globally holomorphically similar on X.* □

Remark 3.6. The statement of this corollary is optimal: the $\mathcal{C}^\infty$-smoothness condition cannot be replaced by a $\mathcal{C}^k$-smoothness with $k < \infty$. See [Lei20, Th. 8.2].

Smith's criterion allows to strengthen Theorem 3.1 as follows.

Corollary 3.7. *Let X be a one dimensional Stein manifold, and let $A_1, A_2 \colon X \to M(n,\mathbb{C})$ be two holomorphic maps. If A_1 and A_2 are locally continuously similar at each point $x \in X$, then they are globally holomorphically similar on X.* □

3.3. **Holomorphically Jordanizable matrices.** Let X be a complex manifold, and $A \colon X \to M(n,\mathbb{C})$ be a holomorphic map.

We say that A is *locally holomoprhically Jordanizable* (or *similar to a Jordan matrix*) at $x_o \in X$ if there exists a neighborhood $U \subseteq X$ of x_o, and a holomorphic map $S \colon U \to GL(n,\mathbb{C})$ such that

$$J(x) = S(x)^{-1} A(x) S(x), \quad x \in U, \tag{3.1}$$

is a Jordan matrix for each $x \in U$. If one can take $U = X$, then we say that A is *globally* holomorphically Jordanizable.

Given $\Phi \in M(n, \mathbb{C})$, denote by $\sigma(\Phi)$ the spectrum of Φ, i.e. the set of its eigenvalues. Furthermore, for $\lambda \in \sigma(\Phi)$ and $k = 1, \dots, n$, denote by $\vartheta_k(\Phi, \lambda)$ the number of Jordan blocks of size k and eigenvalue λ in the Jordan form of Φ, and set

$$\vartheta_k(\Phi) := \sum_{\lambda \in \sigma(\Phi)} \vartheta_k(\Phi, \lambda).$$

Lemma 3.8. *If $A\colon X \to M(n, \mathbb{C})$ is locally holomorphically Jordanizable and (3.1) holds, then we have the following necessary conditions:*

(1) *The Jordan matrix $J(x)$ is holomorphic on some open set $U \subseteq X$,*
(2) *there exist some holomorphic functions $\lambda_1, \dots, \lambda_m\colon U \to \mathbb{C}$ such that $\sigma(A(x)) = \{\lambda_1(x), \dots, \lambda_m(x)\}$ for any $x \in U$,*
(3) *for each $k = 1, \dots, n$, the functions $U \ni x \mapsto \vartheta_k(A(x))$ are constant.*

Proof. Condition (1) directly follows from (3.1). Condition (1) implies both (2) and (3). $\square$

Notice that condition (2) implies that $\operatorname{card} \sigma(A(x)) \leqslant m$ for $x \in U$. In particular, the sign $<$ holds in case of coalescences of some of the eigenvalues $\lambda_1, \dots, \lambda_m$.

Neither condition (1), nor conditions (2) and (3) together, are sufficient for the holomorphic similarity of A.

Counterexample 3.9. Consider the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ in $\mathbb{C}$, and let A and J be the holomorphic matrices given by

$$A(z) = \begin{pmatrix} z & 1 & 0 \\ 0 & z^2 & z \\ 0 & 0 & z^2 \end{pmatrix}, \quad J(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & z^2 & 1 \\ 0 & 0 & z^2 \end{pmatrix}, \quad z \in \mathbb{D}.$$

It is easy to see that $A(z)$ is similar to $J(z)$ for any $z \in \mathbb{D}$. However, $A(z)$ is not holomorphically similar to $J(z)$. Assume there exists a holomorphic matrix $S(z) = (s_{ij}(z))_{i,j=1}^3$ such that $S(z)J(z) = A(z)S(z)$ for $z \in \mathbb{D}$. We have

$$zs_{11} + s_{21} = zs_{11}, \qquad z^2 s_{21} + zs_{31} = zs_{21},$$
$$z^2 s_{22} + zs_{32} = z^2 s_{22}, \qquad z^2 s_{23} + zs_{33} = s_{22} + z^2 s_{23},$$

for $z \in \mathbb{D}$. Hence, $s_{21} = s_{31} = s_{32} = 0$, and $s_{22}(0) = 0 \cdot s_{33}(0) = 0$, so that $S(0)$ is not invertible.

Counterexample 3.10. Conditions (2) and (3) together do not imply condition (1) above. Consider the disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < \frac{1}{2}\}$ in $\mathbb{C}$, and the holomorphic map $A\colon \mathbb{D} \to M(4, \mathbb{C})$ defined by

$$A(z) = \begin{pmatrix} z & 1 & 0 & 0 \\ 0 & -z & 0 & 0 \\ 0 & 0 & 1+z & z \\ 0 & 0 & 0 & 1+z \end{pmatrix}, \quad z \in \mathbb{D}.$$

The eigenvalues of $A(z)$ are given by the holomorphic functions $\lambda_1(z) = z$, $\lambda_2(z) = -z$, $\lambda_3(z) = 1 + z$. Furthermore, we have $\vartheta_1(A(z)) = 2$ and $\vartheta_2(A(z)) = 1$ for each $z \in \mathbb{D}$. The matrix A, however, does not admit a holomorphic Jordan form.

3.4. **Coalescing points.** Let X be a complex manifold, and $A\colon X \to M(n, \mathbb{C})$ be a holomorphic map.

Definition 3.11. A point $x_o \in X$ is called a *coalescing point* of the eigenvalues of A if for any neighborhood U of x_o there exists $x \in U$ such that $\operatorname{card} \sigma(A(x)) > \operatorname{card} \sigma(A(x_o))$. We denote by $\operatorname{coal}(A)$ the set of coalescing points of eigenvalues of A.

For a proof of the following well-known result, see e.g. [Lei17].

Proposition 3.12. *Let $A\colon X \to M(n, \mathbb{C})$ be a holomorphic map, $x_o \in X$, $\sigma(A(x_o)) = \{\lambda_{o,1}, \dots, \lambda_{o,m}\}$, and let n_j be the algebraic multiplicity (i.e. the order as a zero of the characteristic polynomial of $A(x_o)$) of $\lambda_{o,j}$ for $j = 1, \dots, m$.*

The following conditions are equivalent:

(1) *$x_o \notin \operatorname{coal}(A(x_o))$;*
(2) *there exists a neighborhood U of x_o, and uniquely determined holomorphic functions $\lambda_1, \dots, \lambda_m \colon U \to \mathbb{C}$ such that*
   - *$\lambda_j(x_o) = \lambda_{o,j}$ for $j = 1, \dots, m$,*
   - *$\sigma(A(x)) = \{\lambda_1(x), \dots, \lambda_m(x)\}$ for each $x \in U$,*
   - *the eigenvalue $\lambda_j(x)$ of $A(x)$ has algebraic multiplicity n_j for each $x \in U$.* □

Theorem 3.13. *If the set $\operatorname{coal}(A)$ is non-empty, then it is a nowhere dense closed analytic subset of X of codimension 1.* □

Remark 3.14. There exist many sources in the literature for a proof of this result, e.g. see [Bau74, Bau85][FG02, Ch. III, Th. 4.3 and 4.6]. In the recent preprint [Lei17, Th. 4.3], J. Leiterer considered also general complex spaces X (i.e. by allowing singularities). Leiterer's result provides finer estimates: if X is irreducible, and $\operatorname{coal}(A) \neq \emptyset$, then there exist finitely many holomoprhic functions $h_1, \dots, h_\ell \colon X \to \mathbb{C}$ such that

$$\operatorname{coal}(A) = \{x \in X \colon h_j(x) = 0,\ j = 1, \dots, \ell\}.$$

Each h_j is a finite sum of products of the entries of $A(x)$. Moreover, we have

$$|h_j(x)| \leqslant (2n)^{6n^2} \|A(x)\|^{2n^2}, \quad x \in X, \quad 1 \leqslant j \leqslant \ell.$$

3.5. **The gap topology.** In what follows, we denote by $\|f\|$ the operator norm of any linear map $f \colon \mathbb{C}^n \to \mathbb{C}^m$, which is defined by $\|f\| := \sup_{\|x\|=1} \|f(x)\|$. Here the spaces $\mathbb{C}^n, \mathbb{C}^m$ are intended to be equipped with the standard hermitian metric.

Denote by $\mathcal{G}_n$ the set of all $\mathbb{C}$-vector subspaces of $\mathbb{C}^n$. Given $L_1, L_2 \in \mathcal{G}_n$, denote by $\Pi_1, \Pi_2 \colon \mathbb{C}^n \to \mathbb{C}^n$ the orthogonal projections onto L_1 and L_2 respectively. We define the *gap distance* between L_1 and L_2 as

$$\Theta(L_1, L_2) := \|\Pi_1 - \Pi_2\|.$$

Proposition 3.15 ([GLR06, Ch. XIII])**.** *The gap distance Θ defines a metric on $\mathcal{G}_n$. Moreover, we have*

(1) *$\Theta(L_1, L_2) \leqslant 1$ for each $L_1, L_2 \in \mathcal{G}_n$,*

(2) $\Theta(L_1, L_2) < 1$ *only if* $\dim L_1 = \dim L_2$. □

Denote by $G(k,n)$ the Grassmannian of complex k-dimensional subspaces in $\mathbb{C}^n$. The space $G(k,n)$ can be defined as the topological quotient $U(n)/U(k) \times U(n-k)$, with respect to the inclusion $U(k) \times U(n-k) \hookrightarrow U(n)$. In particular, it follows that $G(k,n)$ is compact and connected.

Corollary 3.16. *The connected components of the metric space $(\mathcal{G}_n, \Theta)$ are homeomorphic to the Grassmannians of complex subspaces in $\mathbb{C}^n$, i.e. $\mathcal{G}_n \cong \coprod_{k=0}^n G(k,n)$. In particular, the metric space $(\mathcal{G}_n, \Theta)$ is complete.*

Proof. By the previous proposition, the function $L \mapsto \dim L$ is locally constant on $\mathcal{G}_n$, and consequently constant on the connected components of $\mathcal{G}_n$. Denote $\mathcal{G}_{n,k}$ the connected component of $\mathcal{G}_n$ whose points are k-dimensional subspaces of $\mathbb{C}^n$. We claim that the identity map $G(k,n) \to \mathcal{G}_{n,k}$ is continuous, for each $k = 0, \dots, n$. To see this, it is sufficient to show the continuity of the map $f\colon U(n) \to \mathcal{G}_{n,k}$ defined by

$$A \mapsto \mathbb{C}\text{-span of the first } k \text{ columns of } A.$$

Let $A_1, A_2 \in U(n)$, and denote by $\widetilde{A}_1, \widetilde{A}_2$ the $n \times k$ matrices obtained by the first k columns of A_1 and A_2, respectively. Let $\Pi_1, \Pi_2 \in \mathrm{End}(\mathbb{C}^n)$ be the orthogonal projections onto $f(A_1)$ and $f(A_2)$, respectively. With respect to the standard basis of $\mathbb{C}^n$, the matrices representing Π_1 and Π_2 equal $\widetilde{A}_1\widetilde{A}_1^*$ and $\widetilde{A}_2\widetilde{A}_2^*$. We have

$$\begin{aligned}\Theta(f(A_1), f(A_2)) = \|\Pi_1 - \Pi_2\| &= \|\widetilde{A}_1\widetilde{A}_1^* - \widetilde{A}_2\widetilde{A}_2^*\| = \|\widetilde{A}_1(\widetilde{A}_1^* - \widetilde{A}_2^*) + (\widetilde{A}_1 - \widetilde{A}_2)\widetilde{A}_2^*\| \\ &\leqslant \|\widetilde{A}_1\|\|\widetilde{A}_1^* - \widetilde{A}_2^*\| + \|\widetilde{A}_1 - \widetilde{A}_2\|\|\widetilde{A}_2^*\| \\ &\leqslant \|A_1\|\|A_1^* - A_2^*\| + \|A_1 - A_2\|\|A_2^*\| = 2\|A_1 - A_2\|.\end{aligned}$$

This proves that the identity map $G(k,n) \to \mathcal{G}_{n,k}$ is continuous. Moreover it is also closed, since $G(k,n)$ is compact and $\mathcal{G}_n$ is Hausdorff. □

3.6. **Holomorphic families of subspaces.** The topological space $\mathcal{G}_n$ can be made into a complex manifold in a natural way, due to Corollary 3.16.

We call a *holomorphic family of subspaces of* $\mathbb{C}^n$, parametrized by a complex manifold X, any holomorphic map $f\colon X \to \mathcal{G}_n$. If X is connected, then f takes values in a complex Grassmannian $G(k,n)$ for some $k = 0, \dots, n$.

Remark 3.17. There is a 1-1 correspondence between the following data:

(1) a holomorphic family of subspaces $f\colon X \to G(k,n)$;
(2) a rank k holomorphic subbundle of the trivial bundle $\underline{\mathbb{C}^n} := X \times \mathbb{C}^n$;
(3) a continuous map $f\colon X \to \mathcal{G}_n$ such that, for any $x_o \in X$ there exist a neighborhood U of x_o and holomorphic maps $v_1, \dots, v_k\colon U \to \mathbb{C}^n$ such that $v_1(z), \dots, v_k(z)$ are linearly independent and $f(z) = \mathrm{span}\langle v_1(z), \dots, v_k(z)\rangle$ for each $z \in U$.

Proposition 3.18. *Let X be a locally compact metric space, and $T\colon X \to M(n, \mathbb{C})$ a continuous map. The following conditions are equivalent:*

(1) $\dim\ker T(x)$ *is locally constant on* X*;*
(2) $\dim \mathrm{Im}\, T(x)$ *is locally constant on* X*;*

(3) *the map* $f\colon X \to \mathcal{G}_n$, $x \mapsto \ker T(x)$, *is continuous;*
(4) *the map* $f\colon X \to \mathcal{G}_n$, $x \mapsto \operatorname{Im} T(x)$, *is continuous.*

If moreover X is a complex manifold and T is holomorphic, then the conditions above are equivalent to the following ones:

(5) *the map* $f\colon X \to \mathcal{G}_n$, $x \mapsto \ker T(x)$, *is holomorphic;*
(6) *the map* $f\colon X \to \mathcal{G}_n$, $x \mapsto \operatorname{Im} T(x)$, *is holomorphic.*

Proof. The only non-trivial statements are (1) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (6). The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) can be easily proved by adapting the argument of [GLR06, Prop. 13.6.1]. The implications (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) follow from a result of Ph.G.A. Thijsse, [Thi78, Th. 3.1], proved in the more general case of families of complemented subspaces of Banach spaces. See [Jan88, Prop. 5 and Appendix] for the complete argument, and the reproduction of the proof of Thijsse's result. See also the argument of [GLR06, Prop. 18.1.2] for the case $\dim X = 1$. $\square$

Given a holomorphic map $T\colon X \to M(n, \mathbb{C})$, define $n_0 := \min_{x\in X}\{\dim \ker T(x)\}$. The set $\Upsilon(T)$ of *jump points* of T is the set

$$\Upsilon(T) := \{x \in X\colon \dim \ker T(x) > n_0\}.$$

Proposition 3.19 ([Kab76, Satz 1.1])**.** *The set $\Upsilon(T)$ is an analytic subset of X.* $\square$

For $x \in \Upsilon(T)$, the space $\ker T(x)$ is "too big" if compared to $\ker T(y)$ for $y \notin \Upsilon(T)$ near x. We can however introduce a suitable replacement for the space $\ker T(x)$.

With any holomorphic map $T\colon X \to M(n, \mathbb{C})$, there is a naturally associated morphism of $\mathscr{O}_X$-modules (for simplicity denoted by the same symbol)

$$T\colon \mathscr{O}_X^{\oplus n} \to \mathscr{O}_X^{\oplus n},$$

where $\mathscr{O}_X$ denote the structure sheaf of X. For any $x \in X$ denote by $\mathscr{K}\!er\, T_x$ the stalk of the kernel sheaf $\mathscr{K}\!er\, T$. Define $\mathcal{K}[T; x]$ to be the space of values at x of germs of $\mathbb{C}^n$-valued holomorphic functions in $\mathscr{K}\!er\, T_x$, i.e.

$$\mathcal{K}[T; x] = \{v \in \ker T(x) \mid \exists f_x \in \mathscr{K}\!er\, T_x,\ f_x(x) = v\}$$

Theorem 3.20 ([Kab76, Kab12])**.**

(1) *We have $\mathcal{K}[T; x] \subseteq \ker T(x)$ for any $x \in X$.*
(2) *We have $\mathcal{K}[T; x] = \ker T(x)$ for $x \notin \Upsilon(T)$.*
(3) *There exists an analytic subset $\Sigma \subseteq X$ of codimension at least 2 such that $f\colon X \setminus \Sigma \to \mathcal{G}_n$, $x \mapsto \mathcal{K}[T; x]$, is a holomorphic family of subspaces.* $\square$

3.7. **A generalization of a theorem of Thijsse and Wasow.** As shown by Counterexamples 3.9 and 3.10, conditions (1), (2), (3) of Lemma 3.8 are not sufficient to infer the existence of a holomorphic Jordan form. In this section, we find a further condition which, jointly with condition (1), will ensure the locally holomoprhically Jordanizability of a matrix.

Theorem 3.21. *Let X be a complex manifold, and $A\colon X \to M(n, \mathbb{C})$ be a holomorphic map. The matrix A is locally holomorphically Jordanizable at a point $x_o \in X$ if and only if the following conditions are satisfied:*

(1) *there exists a neighborhood U of x_o and a holomorphic map $J\colon U \to M(n,\mathbb{C})$ such that $J(x)$ is a Jordan form of $A(x)$ for each $x \in U$; in particular, there exist local holomorphic functions $\lambda_1, \dots, \lambda_r\colon U \to \mathbb{C}$ such that $\sigma(A(x)) = \{\lambda_1(x), \dots, \lambda_r(x)\}$ for each $x \in U$;*

(2) *for each $i = 1, \dots, r$ the limits of the generalized eigenspaces*

$$\lim_{\substack{z\to x_o\\ z\notin \operatorname{coal}(A)}} \ker\left(A(z) - \lambda_i(z)\mathrm{Id}\right)^n$$

exist in the gap topology of $\mathcal{G}_n$;

(3) *we have*

$$\mathbb{C}^n = \bigoplus_{i=1}^{r} \lim_{\substack{z\to x_o\\ z\notin \operatorname{coal}(A)}} \ker\left(A(z) - \lambda_i(z)\mathrm{Id}\right)^n .$$

Moreover, conditions (2) and (3) are respectively equivalent to the following ones:

(2bis) *the function $x \mapsto \dim \mathcal{K}[(A - \lambda_i\mathrm{Id})^n; x]$ is continuous at x_o, for any $i = 1, \dots, r$;*

(3bis) *we have $\mathbb{C}^n = \bigoplus_{i=1}^r \mathcal{K}[(A - \lambda_i\mathrm{Id})^n; x_o]$.*

Remark 3.22. If the limit of point (2) exists it necessarily equals $\mathcal{K}[(A - \lambda_i\mathrm{Id})^n; x_o]$. This follows form Proposition 3.18 and Theorem 3.20. This implies the equivalences of (2) and (2bis), and of (3) and (3bis). Moreover, notice that conditions (2), (2bis), (3), and (3bis) are trivially satisfied if $x_o \notin \operatorname{coal}(A)$. This follows from Propositions 3.12 and 3.18.

Before proving the theorem, we consider a simpler case, namely that of a holomorphic Jordan form with one eigenvalue only.

Lemma 3.23. *Let $T\colon X \to M(n,\mathbb{C})$ be a holomorphic map. Assume that*

(1) *there exists a holomorphic map $J\colon X \to M(n,\mathbb{C})$ such that $J(x)$ is a Jordan form of $A(x)$ for each $x \in X$,*

(2) *T has a unique holomorphic eigenvalue function $\lambda\colon X \to \mathbb{C}$.*

Then T is holomorphically Jordanizable on any domain of X biholomorphic to a polydisc.

Proof. For any $z \in X$, set $M(z) := T(z) - \lambda(z)\mathrm{Id}$, and $N_j(z) := \ker M(z)^j$ for any $j = 1, 2, 3, \dots$. We have the tower of subspaces

$$N_1(z) \subseteq N_2(z) \subseteq N_3(z) \subseteq \dots \subseteq N_{n-1}(z) \subseteq N_n(z) = N_{n+1}(z) = \dots .$$

First, we show that we have well-defined holomorphic maps $N_j\colon X \to \mathcal{G}_n$ for any j.

For each j, let us introduce the analytic subsets $\Upsilon_j \subseteq X$ of jump points of M^j. We claim that $\Upsilon_j = \emptyset$ for any j.

For $j \geqslant n$ the statement is obvious. We have the following facts:

(i) since the matrix T has a single eigenvalue λ, we have

$$\vartheta_k(T(z)) = \vartheta_k(T(z), \lambda(z)), \quad z \in X, \quad k = 1, \dots, n;$$

(ii) since $J(z)$ is holomorphic, the function $z \mapsto \vartheta_k(T(z))$ is constant on X for any k;

(iii) we have

$$\vartheta_k(T(z), \lambda(z)) = \dim \frac{N_k(z)}{N_{k-1}(z)} - \dim \frac{N_{k+1}(z)}{N_k(z)}, \quad k = 1, \dots, n, \quad N_0 := 0.$$

From (i), (ii), (iii) for $k = n$, and the fact that $\Upsilon_n = \emptyset$, we deduce that $\Upsilon_{n-1} = \emptyset$. Then, by applying (i), (ii), (iii) for $k = n-1$, we deduce that $\Upsilon_{n-2} = \emptyset$. By iteration of this argument, one proves that all the sets Υ_j are empty. So $N_j \colon X \to \mathcal{G}_n$ are holomorphic for any j. In particular, by Remark 3.17, each N_j can be seen as a holomorphic subbundle of $\underline{\mathbb{C}^n} := X \times \mathbb{C}^n$.

Let $\Delta \subseteq X$ a domain biholomorphic to a polydisc. In particular, Δ is a Stein manifold. For any $j \geqslant 1$, there exists a holomorphic vector subbundle $V_j \to \Delta$ of $\underline{\mathbb{C}^n}|_\Delta$ such that $N_j|_\Delta \oplus V_j = N_{j+1}|_\Delta$. This follows from a general result of Shubin [Shu70, Th. 1]. Since Δ is Stein, the topological and analytical classifications of vector bundles coincide, by the Oka-Grauert principle. Consequently, all vector bundles $N_j|_\Delta$ and V_j, with $j \geqslant 1$, are trivial.

Let n_j be the dimension of the subspaces $N_j(z)$ for any $j \geqslant 1$, and set $\ell := \min\{j \colon n_j = n_{j+1}\}$. Since $V_{\ell-1}$ is trivial, there exist a global basis of sections, i.e. holomorphic maps $v_1, \ldots, v_{n_\ell - n_{\ell-1}} \colon \Delta \to \mathbb{C}^n$ such that

$$N_\ell(z) = N_{\ell-1}(z) \oplus \operatorname{span}\langle v_1(z), \ldots, v_{n_\ell - n_{\ell-1}}(z)\rangle, \quad z \in X.$$

The tuple

$$v_1(z), \ldots,\ v_{n_\ell - n_{\ell-1}}(z),\ M(z)v_1(z), \ldots,\ M(z)v_{n_\ell - n_{\ell-1}}(z)$$

is easily seen to be linearly independent. Hence, by the triviality of $V_{\ell-2}$, there exist vector-valued holomorphic functions

$$v_{n_\ell - n_{\ell-1}+1}, \ldots, v_{n_{\ell-1} - n_{\ell-2}} \colon X \to \mathbb{C}^n,$$

such that

$$\begin{aligned} N_{\ell-1}(z) = N_{\ell-2} \oplus \operatorname{span}\langle M(z)v_1(z), \ldots,\ M(z)v_{n_\ell - n_{\ell-1}}(z)\rangle \\ \oplus \operatorname{span}\langle v_{n_\ell - n_{\ell-1}+1}(z), \ldots, v_{n_{\ell-1} - n_{\ell-2}}(z)\rangle, \quad z \in X. \end{aligned}$$

Proceeding in this way, by applying the standard construction, a family of holomorphic Jordan bases of N_ℓ is obtained. □

Proof of Theorem 3.21. The necessity of condition (1) is clear. The limits

$$\lim_{\substack{z \to x_o \\ z \notin \operatorname{coal}(A)}} \ker\left(J(z) - \lambda_i(z)\mathrm{Id}\right)^n, \quad i = 1, \ldots, r,$$

satisfy conditions (2) and (3). Moreover, if $S \colon U \to GL(n, \mathbb{C})$ is such that $J = S^{-1}AS$, we have

$$\lim_{\substack{z \to x_o \\ z \notin \operatorname{coal}(A)}} \ker\left(A(z) - \lambda_i(z)\mathrm{Id}\right)^n = \lim_{\substack{z \to x_o \\ z \notin \operatorname{coal}(A)}} S(z) \ker\left(J(z) - \lambda_i(z)\mathrm{Id}\right)^n,$$

and conditions (2) and (3) are satisfied as well.

Let us prove the sufficiency of conditions (1), (2), and (3). If condition (1) is satisfied, there exist holomorphic families of subspaces $L_i \colon X \setminus \operatorname{coal}(A) \to \mathcal{G}_n$ defined by $L_i(z) := \ker\left(A(z) - \lambda_i(z)\mathrm{Id}\right)^n$ for $i = 1, \ldots, r$. By Proposition 3.18 and Kaballo's Theorem 3.20, these families can be prolonged to holomorphic families $L_i \colon X \setminus \Sigma_i \to \mathcal{G}_n$, defined by

$$x \mapsto \mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x],$$

on the complement of analytic subspaces $\Sigma_i \subseteq X$ of codimension $\geqslant 2$. By condition (2), we necessarily have $x_o \notin \bigcup_{i=1}^r \Sigma_i$.

Hence, there exist

- a sufficiently small neighborhood U of x_o on which the functions
$$n_i(x) = \dim \mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x], \quad i = 1, \dots, r,$$
are constant,
- holomorphic functions $W_i\colon U \to GL(n_i, \mathbb{C})$, for $i = 1, \dots, r$, with $W_i(x_o) = \mathrm{Id}$,

such that
$$L_i(z) = W_i(z) L_i^o, \quad L_i^o \equiv \mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x_o], \quad i = 1, \dots, r.$$
Moreover, by condition (3), up to shrinking U, we may assume that $\mathbb{C}^n = L_1(z) \oplus \cdots \oplus L_r(z)$ for all $z \in U$. Define $W\colon U \to GL(n, \mathbb{C})$ as the direct sum $W = W_1 \oplus \cdots \oplus W_r$. We have
$$W^{-1}(z) A(z) W(z) = \bigoplus_{i=1}^{r} \widetilde{A}_i(z), \quad z \in U,$$
where each matrix $\widetilde{A}_i(z)$ has a unique eigenvalue $\lambda_i(z)$, for $i = 1, \dots, r$. Moreover, for any $i = 1, \dots, r$, there exist holomorphic matrix-valued functions $\widetilde{J}_i\colon U \to M(n, \mathbb{C})$ which are the Jordan forms of $\widetilde{A}_i(z)$, by condition (1). The result follows by applying Lemma 3.23 to each matrix $\widetilde{A}_i$, with $i = 1, \dots, r$.

This completes the proof. □

Corollary 3.24 ([Thi85, Main Theorem] [Was85, Th. 12.2-2])**.** *Let X be a one dimensional Stein manifold, and $A\colon X \to M(n, \mathbb{C})$ be a holomorphic map. The matrix A is* globally *holomorphically Jordanizable if and only if the following conditions are satisfied:*

1. *there exists a holomorphic map $J\colon X \to M(n, \mathbb{C})$ such that $J(x)$ is a Jordan form of $A(x)$ for each $x \in X$; in particular, there exist holomorphic functions $\lambda_1, \dots, \lambda_r\colon X \to \mathbb{C}$ such that $\sigma(A(x)) = \{\lambda_1(x), \dots, \lambda_r(x)\}$ for each $x \in X$;*
2. *we have $\mathbb{C}^n = \bigoplus_{i=1}^{r} \mathcal{K}[(A - \lambda_i \mathrm{Id})^n; x]$ for each $x \in X$.*

Proof. The result follows from Corollary 3.7, and Theorem 3.21. Notice that conditions (2) and (2bis) of Theorem 3.21 are trivially satisfied in the one dimensional case, due to Kaballo's Theorem 3.20. □

Example 3.25. Consider the holomorphic map $A\colon \mathbb{C}^2 \to M(3, \mathbb{C})$ defined by $A(\boldsymbol{x}) = \begin{pmatrix} x_1 & 0 & x_2 \\ 0 & x_1 & x_2 \\ 0 & 0 & 0 \end{pmatrix}$. There exists holomorphic eigenvalues functions $\lambda_1(\boldsymbol{x}) = x_1$ and $\lambda_2(\boldsymbol{x}) = 0$. We have $\mathrm{coal}(A) = \{x_1 = 0\}$, and the only points at which $A(\boldsymbol{x})$ is not diagonalizable are $\mathrm{coal}(A) \setminus \{(0,0)\}$. The only point $\boldsymbol{x}_o$ at which one of the limits of condition (2) of Theorem 3.21 does not exist is $\boldsymbol{x}_o = (0,0)$. For $\boldsymbol{x} \notin \mathrm{coal}(A)$, we have
$$\ker\left(A(\boldsymbol{x}) - \lambda_2(\boldsymbol{x})\mathrm{Id}\right)^3 = \mathcal{K}[(A - \lambda_2 \mathrm{Id})^3; \boldsymbol{x}] = \mathrm{span}\left\langle (-x_2, -x_2, x_1)^T \right\rangle,$$

whose limit $\boldsymbol{x} \to (0,0)$ does not exist. Notice that at points $\boldsymbol{x}$ in $\mathrm{coal}(A) \setminus \{(0,0)\}$ condition (2) is satisfied, but not condition (3). We have indeed

$$\mathcal{K}[(A-\lambda_1\mathrm{Id})^3\,;\boldsymbol{x}] = \mathrm{span}\big\langle (1,0,0)^T, (0,1,0)^T \big\rangle$$
$$\supseteq \quad \mathcal{K}[(A-\lambda_2\mathrm{Id})^3\,;\boldsymbol{x}] = \mathrm{span}\big\langle (-x_2,-x_2,0)^T \big\rangle.$$

3.8. **Holomorphic Jordanization and bundles of matrices.** Given two multisets $\mathcal{X}_1 = (X_1, m_1)$ and $\mathcal{X}_2 = (X_2, m_2)$, we define their *multiunion* $\mathcal{X}_1 \vee \mathcal{X}_2$ as the multiset $(X_1 \cup X_2, m)$, where the multiplicity function $m\colon X_1 \cup X_2 \to \mathbb{N}^*$ is defined as follows

$$m(x) := \begin{cases} m_1(x), & \text{if } x \in X_1 \setminus X_2, \\ m_2(x), & \text{if } x \in X_2 \setminus X_1, \\ m_1(x) + m_2(x), & \text{if } x \in X_1 \cap X_2. \end{cases}$$

We have a natural "forgetful" surjection $\xi\colon \mathscr{P}(2,n) \to \mathscr{P}(n)$, defined by

$$\{\!\{\lambda_1, \dots, \lambda_r\}\!\} \mapsto \lambda_1 \vee \dots \vee \lambda_r.$$

For each $\lambda \in \mathscr{P}(n)$, define

$$\mathcal{F}_\lambda := \coprod_{\boldsymbol{\nu} \in \xi^{-1}(\lambda)} \mathcal{M}_{\boldsymbol{\nu}}.$$

Theorem 3.26. *Let X be a connected complex manifold, and $A\colon X \to M(n,\mathbb{C})$ be a holomorphic map, locally holomorphically Jordanizable at each point of X. Then we have $A(X) \subseteq \mathcal{F}_\lambda$ for some $\lambda \in \mathscr{P}(n)$.*

Proof. For any $k = 1, \dots, n$, the map $x \mapsto \vartheta_k(A(x))$ is constant on X, by connectedness of X and Lemma 3.8. Assume $x_1, x_2 \in X$ to be such that $A(x_1) \in \mathcal{M}_{\boldsymbol{\lambda}_1}$ and $A(x_1) \in \mathcal{M}_{\boldsymbol{\lambda}_2}$. We have $\vartheta_k(A(x_1)) = \vartheta_k(A(x_2))$ for any $k = 1, \dots, n$. This means that the multiplicities of k as an element of both $\xi(\boldsymbol{\lambda}_1)$, and $\xi(\boldsymbol{\lambda}_2)$ resp., are all equal for any $k = 1, \dots, n$. Hence $\xi(\boldsymbol{\lambda}_1) = \xi(\boldsymbol{\lambda}_2)$. □

Although the inverse statement is not true (as shown, e.g., by Counterexample 3.9), we have the following result.

Theorem 3.27. *Assume $A\colon X \to M(n,\mathbb{C})$ is holomorphic with $A(X) \subseteq \mathcal{M}_{\boldsymbol{\lambda}}$ for some $\boldsymbol{\lambda} \in \mathscr{P}(2,n)$. Then A is locally holomorphically Jordanizable at each point of X.*

Proof. Since $A(X) \subseteq \mathcal{M}_{\boldsymbol{\lambda}}$, we have $\mathrm{coal}(A) = \emptyset$. Hence conditions (2) and (3) of Theorem 3.21 are trivially satisfied. Also condition (1) holds true, by Proposition 3.12. □

4. Universality of integrable deformations of solutions of RHB problems

4.1. **Riemann–Hilbert–Birkhoff problems.** Consider a disc D in $\mathbb{P}^1$, centered at $z = \infty$. Given a holomorphic vector bundle E^o on D, equipped with a meromorphic connection ∇^o admitting a pole at $z = \infty$, the Riemann–Hilbert–Birkhoff (RHB) problem is the following:

Problem 4.1. Does there exist a *trivial* vector bundle E^o on $\mathbb{P}^1$ equipped with a meromorphic connection ∇^o, restricting to the given data on D, and with a further *logarithmic* pole only at $z = 0$?

Assume that the pole at $z=\infty$ is of order 2: in a basis of sections on D, the meromorphic connection has matrix of connection 1-forms $\Omega=-A(z)\mathrm{d}z$, where the $n\times n$ matrix $A(z)$ equals

$$A(z)=\sum_{k=0}^{\infty}A_kz^{-k},\quad A_0\neq 0.$$

Denote by $\mathbb{C}\{z^{-1}\}$ the ring of convergent power series in z^{-1}. The RHB Problem 4.1 is then equivalent to find a so-called *Birkhoff normal form*: does it exist a matrix $G\in GL(n,\mathbb{C}\{z^{-1}\})$ such that $B(z)=G^{-1}AG-G^{-1}\frac{d}{dz}G$ is of the form

$$B(z)=B_0+\frac{B_1}{z},\quad B_0,B_1\in M(n,\mathbb{C})?$$

Remark 4.2. The RHB Problem 4.1 is not always solvable. G. Birkhoff himself proved that the problem is solvable provided that the monodromy matrix of the differential system $\frac{d}{dz}Y(z)=A(z)Y$ is diagonalizable [Bir13], but he seemed to believe that the same would hold generally. This was disproved in 1959 by F.R. Gantmacher and P. Masani: they independently exhibited connections Ω which cannot be put in Birkhoff normal form. See [Gan59] and [Mas59]. The counterexamples found by Gantmacher and Masani are of *reducible* nature, in the sense that they can be put in lower triangularly blocked form via an analytic transformation. This led to the following restricted problem: is the RHB Problem 4.1 solvable in the *irreducible* case? This question was answered positively, first for rank $n=2$ by W.B. Jurkat, D.A. Lutz, and A. Peyerimhoff [JLP76], then for $n=3$ by W.Balser [Bal90], and finally for any dimension by A.A. Bolibruch [Bol94a, Bol94b]. We also refer to [Sab07, Ch.IV], where the reader can find further *irreducibility* assumptions (named after J. Plemelj, A.A. Bolibruch, and V. Kostov) ensuring the solvability of the RHB Problem 4.1. For the reducible case, the reader can find in [BB97] sufficient conditions for the solvability of the problem.

Remark 4.3. The RHB Problem 4.1 admits several variants.

(1) If one allows $B(z)$ to have a more general form

$$B(z)=B_{-N}z^N+\dots+B_{-1}z+B_0+\frac{B_1}{z},\quad N\geqslant 1,\quad B_j\in M(n,\mathbb{C}),$$

then the problem always admits a positive solution. This was the original result proved by G. Birkhoff in [Bir09], see also [Sib90, §3.3].

(2) If one allows *meromorphic* equivalences, i.e. gauge transformations $B(z)=G^{-1}AG-G^{-1}\frac{d}{dz}G$ with $G\in GL(n,\mathbb{C}\{z^{-1}\}[z])$, then the problem is known to be solvable in several cases. For example, if $n=2,3$, then the problem is always solvable as proved by W.B. Jurkat, D.A. Lutz, and A. Peyerimhoff [JLP76], and W. Balser [Bal89]. For arbitrary ranks n, but under the assumption that A_0 has pairwise distinct eigenvalues, H. Turrittin showed that the problem always admits a positive solution [Tur63] [Sib90, §3.10].

4.2. **Families of Riemann–Hilbert–Birkhoff problems.** Throughout the remaining part of the paper, X will denote a connected complex manifold of dimension d. If $Z\subseteq X$ is a smooth analytic hypersurface, we denote by $\mathscr{O}_X(*Z)$ the sheaf of meromorphic functions on

X with poles on Z at most. If E is a holomorphic vector bundle on X, with sheaf of sections $\mathscr{E}$, we set $\mathscr{E}(*Z) := \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X(*Z)$.

In what follows, we want to consider families of RHB problems, parametrized by points of X.

Definition 4.4. Let (E^o, ∇^o) be a holomorphic vector bundle on a disc $D \subseteq \mathbb{P}^1$, centered at $z = \infty$, equipped with a meromorphic connection with a pole of order 2 at $z = \infty$. An *integrable deformation* (∇, E, X, x_o) of (E^o, ∇^o) parametrized by X is the datum of

- a vector bundle E on $D \times X$,
- a *flat* connection ∇ on E with a pole of order 2 along $\{\infty\} \times X$,
- a point $x_o \in X$ at which (E, ∇) restricts to (E^o, ∇^o).

The integrable deformation is called *versal* if any other deformation with base space X' is induced by the previous one via pull-back by a holomorphic map $\varphi\colon (X', x'_o) \to (X, x_o)$. It is *universal* if the germ at x'_o of the base-change φ is uniquely determined.

Assume (E^o, ∇^o) to be extendable to a solution of the RHB Problem 4.1: this means that (in a suitable basis of sections) the matrix of connections 1-forms of ∇^o takes the form

$$\Omega_o = -\left(A_o + \frac{B_o}{z}\right) \mathrm{d}z. \tag{4.1}$$

Let (∇, E, X, x_o) be an integrable deformation of (E^o, ∇^o). The next result shows that, for generic $x \in X$, the restriction $(E, \nabla)|_{D \times x}$ is extendable to a solution of the RHB Problem 4.1, provided that X is simply connected.

Theorem 4.5 ([Sab07, Th. VI.2.1][DH21, Th. 5.1(c)])**.** *Under the assumptions above, if X is simply connected, then there exists*

- *an analytic hypersurface $\Theta \subseteq X \setminus \{x_o\}$,*
- *a* unique *basis of sections of $\mathscr{E}(*(D \times \Theta))$,*

with respect to which the the matrix of connection 1-forms of ∇ takes the form

$$\Omega = -\left(A(x) + \frac{B_o}{z}\right) \mathrm{d}z - z\, C(x), \qquad x \in X \setminus \Theta, \tag{4.2}$$

where

- *the matrix $A(x)$ is a matrix of holomorphic functions on $X \setminus \Theta$, and meromorphic along Θ, such that $A(x_o) = A_o$,*
- *the matrix $C(x)$ is a matrix of holomorphic 1-forms on $X \setminus \Theta$, and meromorphic along Θ, such that $C(x_o) = 0$.* □

Definition 4.6. The matrix-valued holomorphic function $A\colon X \setminus \Theta \to M(n, \mathbb{C})$ above is the *pole part* of the integrable deformation ∇.

The matrix-valued 1-form $C\colon X \setminus \Theta \to M(n, \mathbb{C}) \otimes \Omega^1_X$ above is the *deformation part* of the integrable deformation ∇.

The integrability condition for ∇ translates into the system of equations

$$\mathrm{d}C = 0, \quad C \wedge C = 0, \quad [A, C] = 0, \quad \mathrm{d}A = C + [C, B_o]. \tag{4.3}$$

If $\boldsymbol{x}=(x^1,\dots,x^d)$ are local holomorphic coordinates on X, and if $C=\sum_{i=1}^d C_i(\boldsymbol{x})\mathrm{d}x^i$, these equations take the form

$$\frac{\partial C_i}{\partial x^j}=\frac{\partial C_j}{\partial x^i},\quad [C_i,C_j]=0,\quad [A,C_i]=0,\quad \frac{\partial A}{\partial x^i}=C_i+[C_i,B_o],\tag{4.4}$$

for $i,j=1,\dots,d$.

4.3. **Universal integrable deformations: Malgrange's and Jimbo–Miwa–Ueno's theorems.** Let (E^o,∇^o) be a solution of a RHB problem 4.1, i.e. a trivial vector bundle (of rank n) on $\mathbb{P}^1$ with meromorphic connection with matrix (in a suitable basis of sections) of the form

$$\Omega_o=-\left(A_o+\frac{B_o}{z}\right)\mathrm{d}z.\tag{4.5}$$

Question 1: Under which conditions on (A_o,B_o) there exists a universal integrable deformation of the connection (4.5)?

A first positive result, due to B. Malgrange, requires that the pole part A_o is an element of the regular stratum $\mathcal{M}_{\rm reg}$ of Section 2.6.

Theorem 4.7 ([Mal83a, Mal86])**.** *Assume that the matrix A_o is regular. The connection ∇^o with matrix (4.5) has a germ of universal deformation.*

The reader can find the proof in Appendix A.1.

This result can be made more explicit, under the further *semisimplicity* assumption on A_o. Assume that $A_o\in\mathcal{M}_{\rm reg}\cap\mathcal{M}_{\rm diag}=\mathcal{M}_{\{\!\{1;1;\dots;1\}\!\}}$, and let $P\in GL(n,\mathbb{C})$ such that

$$P^{-1}A_oP=\Lambda_o=\mathrm{diag}(u_o^1,\dots,u_o^n),\quad u_o^i\neq u_o^j,\ \text{for } i\neq j.$$

Set

$$\widehat{\Omega}_o:=P^{-1}\Omega_oP=-\left(\Lambda_o+\frac{\mathcal{B}_o}{z}\right)\mathrm{d}z,\quad \mathcal{B}_o:=P^{-1}B_oP.\tag{4.6}$$

For $\boldsymbol{u}\in\mathbb{C}^n$, denote $\Lambda(\boldsymbol{u}):=\mathrm{diag}(u^1,\dots,u^n)$, so that $\Lambda(\boldsymbol{u}_o)=\Lambda_o$. Given a matrix M denote by M' its diagonal part, and by M'' its off-diagonal part.

Theorem 4.8 ([Mal83b, Mal86])**.** *Under the assumptions above, there exists a sufficiently small polydisc $\mathbb{D}=\mathbb{D}(\boldsymbol{u}_o)\subseteq\mathbb{C}^n$ with center at $\boldsymbol{u}_o$, and a holomorphic off-diagonal matrix $\Gamma\colon\mathbb{D}\to M(n,\mathbb{C})$, $\Gamma(\boldsymbol{u})=\Gamma''(\boldsymbol{u})$, such that:*

(1) *the matrix of 1-forms $\widehat{\Omega}$ on $\mathbb{C}^*\times\mathbb{D}$, defined by*

$$\widehat{\Omega}(z,\boldsymbol{u}):=-\mathrm{d}\left(z\Lambda(\boldsymbol{u})\right)-\left([\Gamma(\boldsymbol{u}),\Lambda(\boldsymbol{u})]+\mathcal{B}_o'\right)\frac{\mathrm{d}z}{z}-[\Gamma(\boldsymbol{u}),\mathrm{d}\Lambda(\boldsymbol{u})],\tag{4.7}$$

defines an integrable connection ∇ on the trivial bundle $\underline{\mathbb{C}^n}\to\mathbb{C}^*\times\mathbb{D}$;

(2) *the $\mathrm{d}z$-component of $\widehat{\Omega}$ restricts to $\widehat{\Omega}_o$ at $\boldsymbol{u}_o$, i.e.*

$$\widehat{\Omega}(z,\boldsymbol{u}_o)=\widehat{\Omega}_o+\omega,\quad \omega\in M(n,\Omega^1_{\mathbb{D}});\tag{4.8}$$

(3) ∇ *is formally equivalent at* $z=\infty$ *to the matrix connection*

$$-\,\mathrm{d}\left(z\Lambda(\boldsymbol{u})\right)-\mathcal{B}'_o\frac{\mathrm{d}z}{z}, \tag{4.9}$$

that is there exists a z^{-1}*-formal base change* $\Phi(z,\boldsymbol{u})=\sum_{k=0}^{\infty}\Phi_k(\boldsymbol{u})z^{-k}$, *with* $\Phi_k\colon\mathbb{D}\to M(n,\mathbb{C})$ *holomorphic and* $\Phi_0(\mathbb{D})\subseteq GL(n,\mathbb{C})$, *such that*

$$\Phi^{-1}\widehat{\Omega}\Phi+\Phi^{-1}\mathrm{d}\Phi=-\mathrm{d}\left(z\Lambda(\boldsymbol{u})\right)-\mathcal{B}'_o\frac{\mathrm{d}z}{z};$$

(4) ∇ *defines a* universal *integrable deformation of its restriction at any point* $\boldsymbol{u}\in\mathbb{D}$.

Moreover, the matrix Γ *is* uniquely *determined by these conditions.* □

Remark 4.9. The integrability condition of ∇ is equivalent to the following equations

$$\mathrm{d}[\Gamma,\mathrm{d}\Lambda]=[\Gamma,\mathrm{d}\Lambda]\wedge[\Gamma,\mathrm{d}\Lambda],\qquad \mathrm{d}[\Gamma,\Lambda]=\left[[\Gamma,\mathrm{d}\Lambda],\mathcal{B}'_o+[\Gamma,\Lambda]\right],$$

called *Darboux–Egoroff equations.* In local coordinates $\boldsymbol{u}$, they read

$$\partial_k\Gamma_{ij}=\Gamma_{ik}\Gamma_{kj},\quad k\neq i,j, \tag{4.10}$$

$$(u^j-u^i)\partial_i\Gamma_{ij}=\sum_{k\neq i,j}(u^k-u^j)\Gamma_{ik}\Gamma_{kj}-(b_j-b_i-1)\Gamma_{ij}, \tag{4.11}$$

$$(u^i-u^j)\partial_j\Gamma_{ij}=\sum_{k\neq i,j}(u^k-u^i)\Gamma_{ik}\Gamma_{kj}-(b_j-b_i-1)\Gamma_{ij}, \tag{4.12}$$

where $\Gamma=(\Gamma_{ij})_{i,j=1}^n$, and $\mathcal{B}'_o=\mathrm{diag}(b_1,\dots,b_n)$.

The statement of Theorem 4.8 ca be further refined to a global one. Let Δ be the union of big diagonal hyperplanes in $\mathbb{C}^n$, defined by the equations

$$\Delta:=\bigcup_{i<j}\{\boldsymbol{u}\in\mathbb{C}^n\colon u^i=u^j\},$$

let X_n be the complement $\mathbb{C}^n\backslash\Delta$, with base point $\boldsymbol{u}_o:=(u_o^1,\dots,u_o^n)$. Denote by $\pi\colon(\widetilde{X}_n,\tilde{\boldsymbol{u}}_o)\to(X_n,\boldsymbol{u}_o)$ the universal cover of X_n, equipped with fixed base points $\tilde{\boldsymbol{u}}_o$ and $\boldsymbol{u}_o$, respectively. The space X_n is identified with the space of diagonal regular $n\times n$ matrices.

Theorem 4.10 ([JMU81, Mal83b])**.** *There exists on* $\mathbb{P}^1\times\widetilde{X}_n$ *a vector bundle* E, *equipped with a meromorphic connection* ∇, *such that*

(1) *the coefficients of* ∇ *have poles along the hypersurface* $\Theta\subseteq\widetilde{X}_n$ *of points* $\tilde{\boldsymbol{u}}\in\widetilde{X}_n$ *such that* $E|_{\mathbb{P}^1\times\{\tilde{\boldsymbol{u}}\}}$ *is not trivial;*

(2) ∇ *is flat, with a pole of Poincaré rank 1 along* $\{\infty\}\times\widetilde{X}_n$, *and a logarithmic pole along* $\{0\}\times\widetilde{X}_n$;

(3) (E,∇) *restricts to* (E^o,∇^o) *at* $\tilde{\boldsymbol{u}}_o$;

(4) *for any* $\tilde{\boldsymbol{u}}\in\widetilde{X}_n$, *the eigenvalues of the pole part of* ∇ *at the point* $(\infty,\tilde{\boldsymbol{u}})$ *equal (up to permutation) the* n*-tuple* $\pi(\tilde{\boldsymbol{u}})$.

Moreover, for any $\tilde{\boldsymbol{u}}\in\widetilde{X}_n\setminus\Theta$, *the bundle with meromorphic connection* (E,∇) *induces a universal deformation of its restriction* $(E,\nabla)|_{\mathbb{P}^1\times\{\tilde{\boldsymbol{u}}\}}$. □

4.4. **Integrable deformations of degenerate Birkhoff normal forms: Sabbah's theorem.** Malgrange's and Jimbo–Miwa–Ueno's Theorems 4.7, 4.8, 4.10 provide an answer to Question 1, in the case the pole part $A_o \in \mathcal{M}_{\rm reg}$. In a sense, these results are the best possible: if $A_o \notin \mathcal{M}_{\rm reg}$, then in general the connection (4.5) does not admit versal deformations, see the example in Appendix A.2.

Consider now the stratum $\mathcal{M}_{\rm diag}$ of diagonalizable matrices, that is the conjugate stratum of $\mathcal{M}_{\rm reg}$ in the sense of Section 2.6.

Let us assume that $A_o \in \mathcal{M}_{\rm diag}$: that is, in the notations of the previous section, assume $\boldsymbol{u}_o \in \Delta$. Define the partition $\{1,\dots,n\} = \coprod_{r\in R} I_r$ such that for any $r \in R$ we have

$$\{i,j\} \subseteq I_r \quad \text{if and only if} \quad u_o^i = u_o^j.$$

In [Sab21], C. Sabbah addressed the following problem.

Question 2: Is it possible to find an integrable deformation of the form (4.7) of the Birkhoff normal form (4.6) with z^{-1}-formal normal form (4.9)?

Remarkably, in [Sab21, Section 4] it is shown that the answer is positive, under (sharp) sufficient conditions on the coefficient $\mathcal{B}_o$ of the normal form (4.6).

Property PNR[9]: There exists a matrix $P \in GL(n,\mathbb{C})$ diagonalizing A_o, i.e. $P^{-1}A_oP = \Lambda_o = \operatorname{diag}(u_o^1,\dots,u_o^n)$, and such that the matrix $\mathcal{B}_o := P^{-1}B_oP$ has the following properties:

($\star$) $\mathcal{B}_o'' \in \operatorname{Im}\operatorname{ad}(\Lambda(\boldsymbol{u}_o))$.

($\star\star$) $\mathcal{B}_o'$ is *partially non-resonant*, i.e.

$$\forall\, r \in R, \quad \forall\, i,j \in I_r, \quad (\mathcal{B}_o')_{ii} - (\mathcal{B}_o')_{jj} \notin \mathbb{Z}\setminus\{0\}.$$

Theorem 4.11 ([Sab21, Th. 4.9])**.** *Assume that Property PNR above holds true. Let $\boldsymbol{u}_o \in \Delta$, and $\mathcal{V}$ a neighborhood of $\boldsymbol{u}_o$ in $\mathbb{C}^n$. If $\mathcal{V}$ is sufficiently small, there exists a holomorphic hypersurface Θ in $\mathcal{V}\setminus\{\boldsymbol{u}_o\}$ and a holomorphic off-diagonal matrix $\Gamma''(\boldsymbol{u})$ on $\mathcal{V}\setminus\Theta$, such that*

1. *the 1-forms matrix (4.7) defines a meromorphic connection ∇ on the trivial vector bundle on $\mathbb{P}^1 \times (\mathcal{V}\setminus\Theta)$;*
2. *∇ restricts to the connection (4.6) at $\boldsymbol{u}_o$;*
3. *∇ is formally equivalent at $z=\infty$ to the connection defined by the matrix of 1-forms (4.9).*

The matrix Γ is uniquely *determined by these conditions.* □

Remark 4.12. Property PNR-($\star$) is equivalent to $(\mathcal{B}_o)_{ij} = 0$ whenever $i,j \in I_r$ for some r. It is a necessary condition for the statement of Theorem 4.11: by restriction of (4.7) at $\boldsymbol{u} = \boldsymbol{u}_o$, we obtain $\mathcal{B}_o'' = [\Gamma(\boldsymbol{u}_o), \Lambda_o]$.

4.5. **Integrable deformations of** d/dv/fs **-type.** Let ∇^o be a connection on a trivial vector bundle $E^o \to \mathbb{P}^1$with matrix of connection 1-forms (4.5). Consider an integrable deformation (∇, E, X, x_o) of ∇^o, parametrized by a complex manifold X, with matrix of connection 1-forms Ω as in equation (4.2). In particular let

$$A\colon X\setminus\Theta \to M(n,\mathbb{C}), \qquad C\colon X\setminus\Theta \to M(n,\mathbb{C})\otimes\Omega^1_X,$$

be the pole and deformation parts, respectively.

[9]PNR stands for "partial non-resonance".

Definition 4.13. The deformation (∇, E, X, x_o) is said to be

- of *diagonal type* (for short, d*-type*) if the pole part A and the deformation part C of Ω are locally holomorphically diagonalizable matrices at x_o;
- of *generic diagonal type* (for short, *generic* d*-type*) if it is of d-type, and if $f_1, \dots, f_n$ are the holomorphic eigenvalues of A, then we have $\mathrm{d}_{x_o} f_i \neq \mathrm{d}_{x_o} f_j$, for any $i \neq j$.

A germ of integrable deformation will be said of (generic) d-type if at least one (and hence any) of its representative is of (generic) d-type. We denote by $\mathfrak{I}_{\mathrm{d}}(\nabla^o)$ (resp. $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$) the classes of germs of integrable deformations of ∇^o which are of d-type (resp. generic d-type).

Remark 4.14. The genericity condition implies that no 1-form $\mathrm{d}(f_i - f_j)$ is vanishing in a neighborhood of x_o. In particular, no pair of eigenvalues f_i and f_j, with $i \neq j$, are identically equal.

Remark 4.15. If (∇, E, X, x_o) is a (germ of) integrable deformation of (generic) d-type, we can construct a new (germ of) integrable deformation $(\nabla', E', X \times \mathbb{C}, (x_o, 0))$ as follows. Set $E' := \mathrm{pr}^* E$, where $\mathrm{pr}\colon X \times \mathbb{C} \to X$. If $\Omega(z, x)$ is the matrix of ∇ as in (4.2), define ∇' by the matrix of connection 1-forms

$$\Omega'(z, x, s) = \Omega(z, x) + \mathrm{d}(zs \cdot \mathrm{Id}_n), \qquad (x, s) \in X \times \mathbb{C}, \quad n = \mathrm{rk}(E).$$

We have $\nabla = \iota^* \nabla'$, where $\iota\colon X \to X \times \mathbb{C}$ is the canonical embedding $\iota(x) = (x, 0)$.

Theorem 4.16. *Let (∇, E, X, x_o) be an integrable deformation of ∇^o of* d*-type. Let $U \subseteq X \setminus \Theta$ a neighborhood of x_o such that $A|_U$, $C|_U$ are holomorphically diagonalizable.*

1. *If $\Delta_0\colon U \to M(n, \mathbb{C})$, with $\Delta_0'' = 0$, is the holomorphic diagonal form of $A|_U$, then $\mathrm{d}\Delta_0 \in \Omega^1_U \otimes M(n, \mathbb{C})$ is the diagonal form of $C|_U$.*
2. *There exists a base of holomorphic sections of $E|_{\mathbb{P}^1 \times U}$ with respect to which ∇ has the following matrix $\widetilde{\Omega}$ of connection 1-forms*
$$\widetilde{\Omega}(z, x) = -\left(\Delta_0(x) + \frac{1}{z}\mathcal{B}(x)\right)\mathrm{d}z - z\,\mathrm{d}\Delta_0(x) + \varpi(x), \tag{4.13}$$
where $\mathcal{B}\colon U \to M(n, \mathbb{C})$ is holomorphic, and $\varpi \in \Omega^1_U \otimes M(n, \mathbb{C})$.

Remark 4.17. The integrability conditions for ∇, in terms of $(\Delta_0, \mathcal{B}, \varpi)$ as in (4.13), read

$$[\mathrm{d}\Delta_0, \mathcal{B}] + [\Delta_0, \varpi] = 0, \qquad \mathrm{d}\mathcal{B} = [\mathcal{B}, \varpi], \qquad \mathrm{d}\Delta_0 \wedge \varpi + \varpi \wedge \mathrm{d}\Delta_0 = 0, \qquad \mathrm{d}\varpi + \varpi \wedge \varpi = 0.$$

Moreover, notice that $\mathcal{B}$ is holomorphically similar to the constant matrix B_o, and hence holomorphically Jordanizable.

Proof. Consider the matrix $\Omega(z, x) = -\left(A(x) + \frac{1}{z}B_o\right)\mathrm{d}z - zC(x)$ defining ∇. The matrices A and C are, by assumption, locally holomorphically diagonalizable at x_o. Moreover we have $[A, C] = 0$, by integrability of ∇. Consequently, there exists a holomorphic map $G\colon U \to GL(n, \mathbb{C})$ which *simultaneously* diagonalizes $A(x)$ and $C(x)$, i.e.

$$G(x)^{-1}A(x)G(x) = \Delta_0(x), \qquad G(x)^{-1}C(x)G(x) = \Delta_1(x).$$

Set $\mathcal{B} := G^{-1}B_oG$, and $\varpi = G^{-1}\mathrm{d}G$. We have

$$G^{-1}\Omega G + G^{-1}\mathrm{d}G = -\left(\Delta_0(x) + \frac{1}{z}\mathcal{B}\right)\mathrm{d}z - z\Delta_1(x) + \varpi,$$

and the integrability conditions read

$$\mathrm{d}\Delta_0 = \Delta_1 + [\Delta_1, \mathcal{B}] + [\Delta_0, \varpi], \qquad \mathrm{d}\mathcal{B} = [\mathcal{B}, \varpi], \qquad [\Delta_0, \Delta_1] = 0,$$
$$\mathrm{d}\Delta_1 + \Delta_1 \wedge \varpi + \varpi \wedge \Delta_1 = 0, \qquad \Delta_1 \wedge \Delta_1 = 0, \qquad \mathrm{d}\varpi + \varpi \wedge \varpi = 0.$$

From the first equation we deduce $\Delta_1 = \mathrm{d}\Delta_0$. □

Definition 4.18. Let (∇, E, X, x_o) be an integrable deformation of d-type. We say that (∇, E, X, x_o) is of *diagonal-vanishing type* (for short, dv-*type*) if there exist

- a neighborhood $U \subseteq X \setminus \Theta$ of x_o,
- a holomorphic off-diagonal matrix $\mathcal{L}\colon U \to M(n, \mathbb{C})$, $\mathcal{L}'' = \mathcal{L}$,
- a basis of holomorphic sections of $E|_{\mathbb{P}^1 \times U}$,

with respect to which ∇ has matrix of connection 1-forms as in (4.13) with

$$\mathcal{B}'' = [\mathcal{L}, \Delta_0], \qquad \varpi'' = [\mathrm{d}\Delta_0, \mathcal{L}]. \tag{4.14}$$

We say that a germ of integrable deformation is of dv-type if at least one (and hence any) of its representative is of dv-type. We denote by $\mathfrak{I}_{\mathrm{dv}}(\nabla^o)$ the class of germs of dv-type integrable deformations of ∇^o.

Theorem 4.19. *Let (∇, E, X, x_o) be an integrable deformation of ∇^o of* dv-*type. There exist a neighborhood U of x_o, and a basis of sections of $E|_{\mathbb{P}^1 \times U}$ with respect to which ∇ has the following matrix of connection 1-forms*

$$\widehat{\Omega}(z, x) = -\left(\Delta_0(x) + \frac{1}{z}\mathfrak{B}(x)\right)\mathrm{d}z - z\,\mathrm{d}\Delta_0(x) + \omega(x), \tag{4.15}$$

where

$$\mathfrak{B}' = \mathfrak{B}'_o = const., \qquad \mathfrak{B}'' = [L, \Delta_0], \qquad \omega' = 0, \qquad \omega'' = [\mathrm{d}\Delta_0, L], \tag{4.16}$$

for some holomorphic off-diagonal matrix $L\colon U \to M(n, \mathbb{C})$.

Proof. Let U be as in Theorem 4.16. Consider a matrix $\widetilde{\Omega}$ defining ∇ as in (4.13). By splitting $\varpi = \varpi' + \varpi''$, we have

$$\mathrm{d}\varpi' = (\varpi'' \wedge \varpi'')' = ([\mathrm{d}\Delta_0, \mathcal{L}] \wedge [\mathrm{d}\Delta_0, \mathcal{L}])' = 0.$$

Since ϖ' is closed, locally there exists an invertible diagonal matrix $H\colon U \to GL(n, \mathbb{C})$ such that $\varpi' = -H^{-1}\mathrm{d}H$. The matrix $\widehat{\Omega} = H^{-1}\widetilde{\Omega}H + H^{-1}\mathrm{d}H$ is as in (4.15), with

$$\mathfrak{B} = H^{-1}\mathcal{B}H, \qquad \omega = H^{-1}\varpi H + H^{-1}\mathrm{d}H.$$

The last three equations of (4.16) are automatically satisfied, with $L = H^{-1}\mathcal{L}H$. The first equation follows from the integrability condition $\mathrm{d}\mathfrak{B} = [\mathfrak{B}, \omega]$: we have

$$\mathrm{d}\mathfrak{B}' = [\mathfrak{B}'', \omega'']' = [[L, \Delta_0], [\mathrm{d}\Delta_0, L]]' = 0.$$ □

Remark 4.20. In terms of the matrices $(\Delta_0, \mathfrak{B}'_o, L)$, the integrability condition of the connection (4.15) reads

$$\mathrm{d}[L, \mathrm{d}\Delta_0] = [L, \mathrm{d}\Delta_0] \wedge [L, \mathrm{d}\Delta_0], \tag{4.17}$$
$$\mathrm{d}[L, \Delta_0] = [[L, \mathrm{d}\Delta_0], \mathfrak{B}'_o + [L, \Delta_0]]\,. \tag{4.18}$$

We call these equations the *generalized Darboux–Egoroff equations*.

In general, the class of dv-type integrable deformations is strictly contained in the class of d-type ones, as the following example shows.

Example 4.21. Let $f\colon \mathbb{C}\to\mathbb{C}$ an arbitrary holomorphic function, and $d_1, d_2\in\mathbb{C}$. Introduce the matrix-valued functions $\Delta_0, \mathfrak{B}\colon \mathbb{C}\to M(2,\mathbb{C})$ and the 1-form valued matrix $\omega\colon \mathbb{C}\to M(2,\mathbb{C})\otimes\Omega^1_{\mathbb{C}}$ defined by

$$\Delta_0(x):=\begin{pmatrix} f(x) & 0\\ 0 & f(x)\end{pmatrix},\quad \mathfrak{B}(x)=\begin{pmatrix} d_1 & (d_1-d_2)x\\ 0 & d_2\end{pmatrix},\quad \omega=\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\mathrm{d}x.$$

The 1-forms valued matrix $\widetilde{\Omega}(z,x):=-\left(\Delta_0(x)+\frac{1}{z}\mathfrak{B}(x)\right)\mathrm{d}z-z\,\mathrm{d}\Delta_0(x)+\omega$ defines an integrable deformation of its restriction at any $x_o\in\mathbb{C}$. Such a deformation is of d-type, but not of dv-type. Assume, indeed, that there exists a gauge equivalence $T(z,x)$ such that $T^{-1}\widetilde{\Omega}T+T^{-1}\mathrm{d}T$ is as in (4.15). In particular, the 1-form $\hat{\omega}:=T^{-1}\omega T+T^{-1}\partial_x T\,\mathrm{d}x$ should satisfy

$$\hat{\omega}=\hat{\omega}'+\hat{\omega}''=[\mathrm{d}\Delta_0, L]=0.$$

Hence, we have $\omega T+\partial T_x\,\mathrm{d}x=0$. If we set

$$T=\begin{pmatrix} T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix},$$

we necessarily have

$$\partial_x T_{11}+T_{21}=0,\qquad \partial_x T_{12}+T_{22}=0,\qquad \partial_x T_{21}=0,\qquad \partial_x T_{22}=0.$$

This implies that

$$T(z,x)=\begin{pmatrix} -xT_{21}(z) & -xT_{22}(z)\\ T_{21}(z) & T_{22}(z)\end{pmatrix}\qquad\Longrightarrow\qquad \det T\equiv 0.$$

This is absurd.

Definition 4.22. Let (∇, E, X, x_o) be an integrable deformation of ∇^o with matrix Ω as in (4.2). We say that (∇, E, X, x_o) is of *formally simplifiable type* (for short, fs*-type*) if there exist

- a neighborhood $U\subseteq X\setminus\Theta$ of x_o,
- a sequence of holomorphic maps $\Phi_k\colon U\to M(n,\mathbb{C})$, with $k\geqslant 0$ and $\Phi_0(U)\subseteq GL(n,\mathbb{C})$,
- a holomorphic diagonal map $\Delta_0\colon U\to M(n,\mathbb{C})$,
- a constant diagonal matrix $\mathfrak{B}'_o\in M(n,\mathbb{C})$,

such that

$$\Phi^{-1}\Omega\Phi+\Phi^{-1}\mathrm{d}\Phi=-\mathrm{d}\left(z\Delta_0(x)\right)-\mathfrak{B}'_o\frac{\mathrm{d}z}{z}.$$

We say that a germ of integrable deformation is of fs-type if at least one (and hence any) of its representative is of fs-type. We denote by $\mathfrak{I}_{\mathrm{fs}}(\nabla^o)$ the class of germs of fs-type integrable deformations of ∇^o.

Remark 4.23. The functions Φ_k in the definition of integrable deformation of fs-type satisfy the following equations:

$$\begin{aligned} &A\Phi_0 = \Phi_0\Delta_0, \qquad C\Phi_0 = \Phi_0 \mathrm{d}\Delta_0, && \\ &A\Phi_{k+1} + B_o\Phi_k + k\Phi_k = \Phi_{k+1}\Delta_0 + \Phi_k\mathfrak{B}'_o, && k \geqslant 0, \\ &\mathrm{d}\Phi_k = C\Phi_{k+1} - \Phi_{k+1}\mathrm{d}\Delta_0, && k \geqslant 0. \end{aligned}$$

Denote by $\mathfrak{I}(\nabla^o)$ the set of all germs of integrable deformations of a connection ∇^o.

Theorem 4.24.

(1) *The classes $\mathfrak{I}_{\rm d}(\nabla^o), \mathfrak{I}_{\rm dv}(\nabla^o)$, and $\mathfrak{I}_{\rm fs}(\nabla^o)$ are closed by arbitrary base change of the deformation parameter spaces $\varphi\colon (X, x_o) \to (X', x'_o)$.*

(2) *For any connection ∇^o of the form (4.5), we have*

$$\begin{tikzcd}[column sep=small, row sep=small] \mathfrak{I}_{\rm fs}(\nabla^o) \arrow[dr, phantom, sloped, "\subseteq"] & \\ & \mathfrak{I}_{\rm dv}(\nabla^o) \subseteq \mathfrak{I}_{\rm d}(\nabla^o) \subseteq \mathfrak{I}(\nabla^o) \\ \mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o) \arrow[ur, phantom, sloped, "\subseteq"] & \end{tikzcd}$$

(3) *If the pole part A_o of ∇^o is in $\mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$, then we have*

$$\mathfrak{I}_{\rm fs}(\nabla^o) = \mathfrak{I}_{\rm dv}(\nabla^o) = \mathfrak{I}_{\rm d}(\nabla^o) = \mathfrak{I}(\nabla^o).$$

Proof. Point (1) is obvious, by definition of the classes $\mathfrak{I}_{\rm d}(\nabla^o), \mathfrak{I}_{\rm dv}(\nabla^o)$, and $\mathfrak{I}_{\rm fs}(\nabla^o)$.

The only nontrivial inclusion of point (2) is $(\mathfrak{I}_{\rm fs}(\nabla^o) \cup \mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o)) \subseteq \mathfrak{I}_{\rm dv}(\nabla^o)$. Let $\nabla \in \mathfrak{I}_{\rm fs}(\nabla^o)$ be defined by the matrix Ω as in (4.2). Let $(\Phi_k)_{k\geqslant 0}$ be the matrix-valued functions such that $\Phi := \sum_{k\geqslant 0} \Phi_k z^{-k}$ satisfies $\Phi^{-1}\Omega\Phi + \Phi^{-1}\mathrm{d}\Phi = -\mathrm{d}\left(z\Delta_0(x)\right) - \mathfrak{B}'_o\frac{\mathrm{d}z}{z}$. By invoking the equations of Remark 4.23, it is easy to see that

$$\Phi_0^{-1}\Omega\Phi_0 + \Phi_0^{-1}\mathrm{d}\Phi_0 = -\left(\Delta_0 + \frac{1}{z}(\mathfrak{B}'_o + [\Phi_0^{-1}\Phi_1, \Delta_0])\right)\mathrm{d}z - z\mathrm{d}\Delta_0 - [\Phi_0^{-1}\Phi_1, \mathrm{d}\Delta_0].$$

This proves that $\mathfrak{I}_{\rm fs}(\nabla^o) \subseteq \mathfrak{I}_{\rm dv}(\nabla^o)$.

Let $\nabla \in \mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o)$. By Theorem 4.16, it can be defined by a matrix $\widetilde{\Omega}$ of connection 1-forms as in (4.13) for a suitable pair $(\mathcal{B}, \varpi)$ of matrices. By integrability, we have $\mathrm{d}\Delta_0 \wedge \varpi + \varpi \wedge \mathrm{d}\Delta_0 = 0$. The equation for the (i, j) entry, with $i \neq j$, reads $\varpi_{ij} \wedge (\mathrm{d}f_i - \mathrm{d}f_j) = 0$. Assume temporarily that the deformation is parametrized by a complex manifold X with $\dim_{\mathbb{C}} X \geqslant 2$. Since $\mathrm{d}f_i - \mathrm{d}f_j \not\equiv 0$ in a neighborhood U of x_o, we deduce the existence[10] of a holomorphic function $\mathcal{L}_{ij}\colon U \to \mathbb{C}$ such that $\varpi_{ij} = \mathcal{L}_{ij}(\mathrm{d}f_i - \mathrm{d}f_j)$. This is a special case of de Rham's division lemma, see [dR54][NY04, Lemma 3.1]. By integrability, we also have $[\mathrm{d}\Delta_0, \mathcal{B}] + [\Delta_0, \varpi] = 0$. The equation for the (i, j) entry reads $(\mathrm{d}f_i - \mathrm{d}f_j)\mathcal{B}_{ij} + (f_i - f_j)\varpi_{ij} = 0$. Hence, we necessarily have $\mathcal{B}_{ij} + (f_i - f_j)\mathcal{L}_{ij} = 0$. This shows that, if $\dim_{\mathbb{C}} X \geqslant 2$, ∇ is of dv-type, since $\mathcal{B}'' = [\mathcal{L}, \Delta_0]$ and $\varpi = [\mathrm{d}\Delta_0, \mathcal{L}]$. If $\dim_{\mathbb{C}} X = 1$, by Remark 4.15, we can consider an extended integrable deformation ∇' parametrized by $X \times \mathbb{C}$ and such that $\nabla = \iota^*\nabla'$, where $\iota\colon X \to X \times \mathbb{C}$, $\iota(x) := (x, 0)$. The connection ∇' is of generic d-type, so that $\nabla' \in \mathfrak{I}_{\rm dv}(\nabla^o)$. The result for ∇ follows from point (1). This proves point (2).

[10]If $\dim_{\mathbb{C}} X = 1$, the vanishing condition $\varpi_{ij} \wedge (\mathrm{d}f_i - \mathrm{d}f_j) = 0$ does not imply that ϖ_{ij} is a multiple of $(\mathrm{d}f_i - \mathrm{d}f_j)$. The vanishing condition is indeed satisfied by any arbitrary holomorphic 1-form ϖ_{ij}.

Let us prove point (3). Consider an arbitrary integrable deformation defined by a matrix Ω as in (4.2). The set $\mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$ is an open dense subset of $M(n,\mathbb{C})$. Hence, if $A_o \in \mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$, there exists a neighborhood $U \subseteq X$ of x_o such that $A(x) \in \mathcal{M}_{\rm reg} \cap \mathcal{M}_{\rm diag}$ for any $x \in U$. Consequently, $A(x)$ is locally holomorphically diagonalizable by Theorem 3.27. By integrability, the matrices $A(x)$ and $C(x)$ commute for any $x \in U$. By regularity of $A(x)$, it follows that $C(x)$ is a polynomial expression of $A(x)$. Thus, $C(x)$ is locally holomorphically diagonalizable. This shows that $\mathfrak{I}(\nabla^o) = \mathfrak{I}_{\rm d}(\nabla^o)$.

Consider an integrable deformation ∇ of d-type, defined by a matrix $\widetilde{\Omega}$ as in (4.13) for a suitable pair $(\mathcal{B}, \varpi)$. By integrability, we have $[{\rm d}\Delta_0, \mathcal{B}] + [\Delta_0, \varpi] = 0$. The equation for the (i,j) entry, with $i \neq j$, reads $({\rm d}f_i - {\rm d}f_j)\mathcal{B}_{ij} + (f_i - f_j)\varpi_{ij} = 0$. By the argument above, we also have $f_i(x) \neq f_j(x)$ for any $i \neq j$ and any $x \in U$. Hence, if we set $\mathcal{L}_{ij} := \frac{\mathcal{B}_{ij}}{f_j - f_i}$, we obatin a holomorphic off-diagonal matrix $\mathcal{L}\colon U \to M(n,\mathbb{C})$ such that $\mathcal{B} = [\mathcal{L}, \Delta_0]$ and $\varpi = [{\rm d}\Delta_0, \mathcal{L}]$. This shows that $\mathfrak{I}_{\rm d}(\nabla^o) = \mathfrak{I}_{\rm dv}(\nabla^o)$.

It thus remains to show that $\mathfrak{I}_{\rm dv}(\nabla^o) \subseteq \mathfrak{I}_{\rm fs}(\nabla^o)$. Let $\nabla \in \mathfrak{I}_{\rm dv}(\nabla^o)$ to be defined by the matrix $\widehat{\Omega}$ as in (4.15). As before, we can take U sufficiently small so that $f_i(x) \neq f_j(x)$ for any $i \neq j$ and any $x \in U$. We claim that there exists a unique sequence of holomorphic functions $F_k\colon U \to M(n,\mathbb{C})$, with $k \geqslant 1$, such that

$$F^{-1}\widehat{\Omega}F + F^{-1}{\rm d}F = -{\rm d}\left(z\Delta_0(x)\right) - \mathfrak{B}'_o\frac{{\rm d}z}{z},$$

where

$$F(z,x) := {\rm Id}_n + \sum_{k=1}^{\infty} F_k(x) z^{-k}, \qquad \mathfrak{B}'_o = \mathfrak{B}(x_o)'.$$

This holds if and only if the following identities are satisfied

$$[\Delta_0, F_{k+1}] + \mathfrak{B}F_k - F_k\mathfrak{B}'_o + kF_k = 0, \qquad k \geqslant 0, \quad F_0 = {\rm Id}_n, \tag{4.19}$$

$${\rm d}F_k = [F_1, {\rm d}\Delta_0]F_k + [{\rm d}\Delta_0, F_{k+1}], \qquad k \geqslant 0. \tag{4.20}$$

The functions F_k's can be found by iteratively solving equations (4.19). For $i \neq j$, we find

$$(F_{k+1})_{ij} = \frac{1}{f_j - f_i}\left(\mathfrak{B}F_k - F_k\mathfrak{B}'_o + kF_k\right)_{ij}, \qquad k \geqslant 0$$

and the diagonal elements are given by

$$(F_{k+1})_{ii} = -\frac{1}{k+1}\sum_{\ell \neq i} \mathfrak{B}_{i\ell}(F_{k+1})_{\ell i}, \qquad k \geqslant 0.$$

The resulting functions F_k's are holomorphic on U. In particular, notice that $F''_1 = L$. Let us prove that equations (4.20) are automatically satisfied. The proof is standard, but for completeness we outline it below. Fix the sector $\mathcal{V}$ in the universal cover of $\mathbb{C}^*$ defined by $\mathcal{V} = \{z \in \widetilde{\mathbb{C}^*}\colon |\arg z| < \frac{\pi}{2}\}$. By a theorem of Y. Sibuya [Sib62, Main Th.] [HS66] [Was65, BJL79], the differential system of equations

$$\frac{d}{dz}Y = \left(\Delta_0(x) + \frac{1}{z}\mathfrak{B}(x)\right)Y \tag{4.21}$$

admits a unique fundamental system of solutions $Y(z,x)$ such that

$$Y(z,x)z^{-\mathfrak{B}'_o}e^{-\Delta_0(x)}\sim F(z,x),\quad |z|\to\infty,\, z\in\mathcal{V},\quad \text{uniformly in } x. \tag{4.22}$$

If we prove that Y satisfies the system of equations

$$\mathrm{d}'Y=(z\,\mathrm{d}'\Delta_0(x)+[F_1,\mathrm{d}'\Delta_0])\,Y,\quad \mathrm{d}':=\sum_{i=1}^{d}\frac{\partial}{\partial x^i}\mathrm{d}x^i,\quad x^i\text{'s coordinates on } X, \tag{4.23}$$

then equations (4.20) immediately follow from the asymptotic expansion (4.22). For any $i=1,\dots,d$, set

$$W_i(z,x):=\partial_i Y(z,x)-(z\partial_i\Delta_0(x)+[F_1,\partial_i\Delta_0])\,Y(z,x).$$

We claim that each function W_i is a solution of (4.21): by a simple computation, and by invoking the first two identities of Remark 4.17, we find that

$$\partial_z W_i-\left(\Delta_0+\frac{1}{z}\mathfrak{B}\right)W_i=\partial_i\left(\partial_z Y-\left(\Delta_0+\frac{1}{z}\mathfrak{B}\right)Y\right)=0.$$

Hence, for any $x\in X$, there exists a matrix $C_i(x)$ such that $W_i(z,x)=Y(z,x)C_i(x)$. On the one hand, we have

$$W_i(z,x)\sim(\partial_i F+z[F,\partial_i\Delta_0]+[F_1,\partial_i\Delta_0]F)\,z^{\mathfrak{B}'_o}e^{z\Delta_0},\quad |z|\to\infty,\, z\in\mathcal{V},\quad \text{uniformly in } x.$$

On the other hand, we also have

$$W_i(z,x)\sim F(z,x)z^{\mathfrak{B}'_o}e^{z\Delta_0}C_i(x),\quad |z|\to\infty,\, z\in\mathcal{V},\quad \text{uniformly in } x.$$

Consequently, we deduce that

$$z^{\mathfrak{B}'_o}e^{z\Delta_0}C_i(x)e^{-z\Delta_0}z^{-\mathfrak{B}'_o}=\text{formal power series in }\frac{1}{z}. \tag{4.24}$$

For a fixed $x\in X$, and for $j\neq k$, the sector $\mathcal{V}$ contains rays ℓ of points z along which $\mathrm{Re}(z(f_j(x)-f_k(x)))>0$. We deduce that the (j,k)-entry of $C_i(x)$ vanishes, otherwise we would have a divergence, for $|z|\to\infty$ along the rays ℓ, on the l.h.s. of (4.24). So the matrix $C_i(x)$ is diagonal, and we have

$$\begin{aligned}C_i(x)&=z^{\mathfrak{B}'_o}e^{z\Delta_0}C_i(x)e^{-z\Delta_0}z^{-\mathfrak{B}'_o}\\&=F(z,x)^{-1}\left(\partial_i F(z,x)+z[F(z,x),\partial_i\Delta_0(x)]-[F_1(x),\partial_i\Delta_0(x)]F(z,x)\right)\\&=z(\partial_i\Delta_0-\partial_i\Delta_0)+(F_1\,\partial_i\Delta_0-\partial_i\Delta_0\,F_1-[F_1,\partial_i\Delta_0])+O\left(\frac{1}{z}\right)=O\left(\frac{1}{z}\right).\end{aligned}$$

This shows that $C_i(x)=0$ for any $i=1,\dots,d$. Hence, (4.23) hold true. This completes the proof. ☐

The following result further clarifies the relation between the classes $\mathfrak{I}_{\mathrm{fs}}(\nabla^o)$ and $\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)$.

Theorem 4.25. *Let ∇^o be a connection with matrix (4.5). Assume one of the following assumptions hold:*

- *The pole part A_o is an element of $\mathcal{M}_{\mathrm{reg}}\cap\mathcal{M}_{\mathrm{diag}}$;*
- *The pole part A_o is an element of $\mathcal{M}_{\mathrm{diag}}$ and the Property PNR holds.*

Then we have $\emptyset\neq\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)\subsetneq\mathfrak{I}_{\mathrm{fs}}(\nabla^o)$.

Proof. Let us first assume that $A_o \in \mathcal{M}_{\mathrm{diag}} \cap \mathcal{M}_{\mathrm{reg}}$. We have $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o) \subseteq \mathfrak{I}_{\mathrm{fs}}(\nabla^o)$, by point (3) of Theorem 4.24. Consider the universal integrable deformation $(\nabla^{\mathrm{JMUM}}, \underline{\mathbb{C}^n}, \mathbb{D}(\boldsymbol{u}_o), \boldsymbol{u}_o)$ of ∇^o, whose existence is established by Malgrange's and Jimbo–Miwa–Ueno's Theorems 4.7 and 4.10. Its germ is an element of $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$. Moreover, if $\Omega_o(z)$ is the matrix of connection 1-forms defining ∇^o as in (4.5), then the matrix $\Omega(z,s) := \Omega_o(z) + \mathrm{d}(zs \cdot \mathrm{Id}_n)$, with $s \in \mathbb{C}$, defines an element of $\mathfrak{I}_{\mathrm{fs}}(\nabla^o) \setminus \mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$. This proves the statement in the case A_o is in $\mathcal{M}_{\mathrm{reg}} \cap \mathcal{M}_{\mathrm{diag}}$.

Let us now assume that $A \in \mathcal{M}_{\mathrm{diag}}$, and that the Property PNR holds. Consider the integrable deformation $(\nabla^{\mathrm{Sab}}, \underline{\mathbb{C}^n}, \mathbb{D}(\boldsymbol{u}_o), \boldsymbol{u}_o)$ of ∇^o, whose existence is established by Sabbah's Theorem 4.11. It defines an element of $\mathfrak{I}_{\mathrm{fs}}(\nabla^o) \cap \mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$. This proves that $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o) \neq \emptyset$.

Assume that $u_o^a = u_o^b$, with $a \neq b$, and consider the hyperplane $\Delta_{ab} := \{u^a = u^b\}$ of $\mathbb{C}^n$. Denote by $\iota\colon \Delta_{ab} \cap \mathbb{D}(\boldsymbol{u}_o) \to \mathbb{D}(\boldsymbol{u}_o)$ the natural inclusion map. The germ of the integrable deformation $(\iota^*\nabla^{\mathrm{Sab}}, \iota^*\underline{\mathbb{C}^n}, \Delta_{ab} \cap \mathbb{D}(\boldsymbol{u}_o), \boldsymbol{u}_o)$ defines an element of $\mathfrak{I}_{\mathrm{fs}}(\nabla^o) \setminus \mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$. Hence, we have $\mathfrak{I}_{\mathrm{fs}}(\nabla^o) \setminus \mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o) \neq \emptyset$.

It remains to show that $\mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o) \subseteq \mathfrak{I}_{\mathrm{fs}}(\nabla^o)$. Consider a connection ∇^o with pole part $A_o \in \mathcal{M}_{\mathrm{diag}}$, and assume that Property PNR holds. Let $(\nabla, E, X, x_o) \in \mathfrak{I}_{\mathrm{d}}^{\mathrm{gen}}(\nabla^o)$ be defined by a matrix $\widehat{\Omega}(z,x)$ as in (4.15), for a suitable triple of holomorphic matrices $(\Delta_0, \mathfrak{B}'_o, L)$ defined on an open neighborhood U of x_o. Without loss of generality, by the Property PNR we may assume that the constant matrix $\mathfrak{B}'_o = \mathrm{diag}(b_1, \dots, b_n)$ itself is such that

$$b_i - b_j \notin \mathbb{Z} \setminus \{0\} \quad \text{whenever } f_i(x_o) = f_j(x_o), \quad \text{for some } i \neq j.$$

One can always recover this condition up to a gauge equivalence $\widehat{\Omega} \mapsto T^{-1}\widehat{\Omega}T$ by a constant matrix T.

We need to show the existence of a sequence of holomorphic matrices $F_k\colon U \to M(n, \mathbb{C})$, with $k \geqslant 1$, such that

$$F^{-1}\widehat{\Omega}F + F^{-1}\mathrm{d}F = -\mathrm{d}\left(z\Delta_0(x)\right) - \mathfrak{B}'_o \frac{\mathrm{d}z}{z}, \qquad F := \mathrm{Id}_n + \sum_{k=1}^{\infty} F_k z^{-k}.$$

As in the proof of point (3) of Theorem 4.24, the functions F_k's can be found by iteratively solving the equations

$$[\Delta_0, F_{k+1}] + [\mathfrak{B}'_o, F_k] + [L, \Delta_0]F_k + kF_k = 0, \qquad k \geqslant 0, \quad F_0 = \mathrm{Id}_n. \tag{4.25}$$

The procedure is standard. The matrices $F_k(x)$ can be computed, entry by entry, in terms of the entries of $F_h(x)$ with $h < k$, for any $x \in U$. At each $x \in U$, the procedure works case by case, according weather $f_i(x) \neq f_j(x)$ or $f_i(x) = f_j(x)$:

- if $f_i(x) \neq f_j(x)$, with $i \neq j$, then

$$F_{k+1}(x)_{ij} = \frac{1}{f_j(x) - f_i(x)} \left([\mathfrak{B}'_o, F_k(x)] + [L(x), \Delta_0(x)]F_k(x) + kF_k(x)\right)_{ij}, \quad k \geqslant 0; \tag{4.26}$$

- if $f_i(x) = f_j(x)$, with $i \neq j$, then

$$F_{k+1}(x)_{ij} = -\frac{1}{b_i - b_j + k + 1} \sum_{\ell} (f_\ell(x) - f_i(x)) L(x)_{i\ell} F_{k+1}(x)_{\ell j}, \qquad k \geqslant 0; \tag{4.27}$$

- the diagonal entries are given by

$$F_{k+1}(x)_{ii} = -\frac{1}{k+1}\sum_{\ell\neq i}\mathfrak{B}(x)_{i\ell}(F_{k+1})_{\ell i}, \qquad k\geqslant 0. \tag{4.28}$$

From this construction, it is clear that the F_k's are holomorphic on $U\setminus \mathrm{coal}(\Delta_0)$. Moreover, on the complement $U\setminus\bigcup_{a,b}\{z\colon f_a(x)=f_b(x)\}$, the following identities hold: for any pair (i,j), with $i\neq j$, we have

$$(\mathrm{d}f_j-\mathrm{d}f_i)(F_{k+1})_{ij} = \left([F_1,\mathrm{d}\Delta_0]F_k-\mathrm{d}F_k\right)_{ij}. \tag{4.29}$$

This follows from the argument used in the proof of point (3) of Theorem 4.24.

Notice that equation (4.25), specialized for $k=0$, is trivially solved by the choice $F_1''=L$. In particular, F_1'' holomorphically extends on the whole open set U, including the coalescence set $\mathrm{coal}(\Delta_0)$. Moreover, equation (4.28) implies that F_1 holomorphically extends to the whole U.

We claim that the holomorphicity of F_1 at a point $x_c\in\mathrm{coal}(\Delta_0)$ implies that *all* the matrices F_k are holomorphic at x_c. In order to prove this, we proceed by induction on k. For $k=1$, the claim is tautological. Assume that $F_1,\dots,F_k$ are holomorphic on X. If we introduce local coordinates $(x^1,\dots,x^d)$ on X, with $d=\dim X$, by assumption there exists $h\in\{1,\dots,d\}$ such that $\partial_h f_j-\partial_h f_i$ is not vanishing on U. Hence, we have

$$(F_{k+1})_{ij} = \frac{1}{\partial_h f_j-\partial_h f_i}\left([F_1,\partial_h\Delta_0]F_k-\partial_h F_k\right)_{ij},$$

and the right-hand-side is holomorphic at x_c. This shows that the off-diagonal entries of each F_k's are holomorphic at x_c. The diagonal entries of F_{k+1} are determined by the off-diagonal ones, by equation (4.28), and they are holomorphic at x_c. This completes the proof. □

Remark 4.26. Due to equations (4.26) and (4.27), the holomorphicity of F_1 and F_2 at points of $\mathrm{coal}(\Delta_0)$ is equivalent to the following condition on the holomorphic matrix L: for any $x_c\in\mathrm{coal}(\Delta_0)$, such that $f_i(x)=f_j(x)$ for some $i\neq j$, there exists a neighborhood $\mathcal{U}(x_c)$ where

$$(b_j-b_i-1)L_{ij}(x)-\sum_{\ell\neq i}(f_\ell(x)-f_i(x))L_{i\ell}(x)L_{\ell j}(x) = O\left(f_i(x)-f_j(x)\right), \quad x\in\mathcal{U}(x_c).$$

At this point of the presentation, it is not not obvious why such an estimate holds true. We will give a direct justification of this fact in the subsequent sections, standing on a deeper analysis of the generalized Darboux–Egoroff system of equations (4.17) and (4.18). See Remark 4.32.

4.6. **Generalized Darboux–Egoroff equations, and its initial value property.** Consider n holomorphic functions $f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x})$ in d complex variables $\boldsymbol{x}=(x^1,\dots,x^d)\in\mathbb{C}^d$. Let $b_1,\dots,b_n\in\mathbb{C}$ be arbitrary constants. In what follows we set $\partial_i=\frac{\partial}{\partial x^i}$.

The *generalized Darboux–Egoroff* system $\mathrm{DE}_{d,n}((f_i)_{i=1}^n;(b_i)_{i=1}^n)$ is the following system of PDE's for n^2-n unknown functions $(F_{kh}(\boldsymbol{x}))_{k,h=1}^n$, with $k\neq h$:

$$(\partial_j f_h-\partial_j f_k)\partial_i F_{kh}-(\partial_i f_h-\partial_i f_k)\partial_j F_{kh}=\\ \sum_{\ell=1}^n(\partial_i f_\ell-\partial_i f_k)(\partial_j f_h-\partial_j f_\ell)F_{k\ell}F_{\ell h}-\sum_{\ell=1}^n(\partial_j f_\ell-\partial_j f_k)(\partial_i f_h-\partial_i f_\ell)F_{k\ell}F_{\ell h}, \quad (4.30)$$

$$(f_h-f_k)\partial_i F_{kh}=(b_h-b_k-1)(\partial_i f_h-\partial_i f_k)F_{kh}\\ +\sum_{\ell=1}^n(\partial_i f_\ell-\partial_i f_k)(f_h-f_\ell)F_{k\ell}F_{\ell h}-\sum_{\ell=1}^n(f_\ell-f_k)(\partial_i f_h-\partial_i f_\ell)F_{k\ell}F_{\ell h}, \quad (4.31)$$

for any $i,j=1,\dots,d$, and any $k,h=1,\dots,n$, with $k\neq h$. Solutions F of the generalized Darboux–Egoroff system can be arranged in a off-diagonal matrix in $M(n,\mathbb{C})$. In matrix notation, equations (4.30), (4.31) read

$$\mathrm{d}[F,\mathrm{d}\Delta_0]=[F,\mathrm{d}\Delta_0]\wedge[F,\mathrm{d}\Delta_0],\qquad \mathrm{d}[F,\Delta_0]=\left[[F,\mathrm{d}\Delta_0],\mathfrak{B}_o'+[F,\Delta_0]\right],$$

where $\Delta_0(\boldsymbol{x})=(f_1(\boldsymbol{x}),\dots,f_n(\boldsymbol{x}))$, and $\mathfrak{B}_o'=\mathrm{diag}(b_1,\dots,b_n)$.

Remark 4.27. In the very special case $d=n=3$, and $f_i(x^1,x^2,x^3)=x^i$, with $i=1,2,3$, the system of nonlinear PDEs (4.30) and (4.31) reduces to the generic Painlevé equation $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$. See [Lor14, Th. 4.1].

Set $\mathcal{A}:=M(n,\mathbb{C})[\![(x^i-x_o^i)_{i=1}^d]\!]$.

Theorem 4.28. *Let $\boldsymbol{x}_o\in\mathbb{C}^d$, and assume that*

- *for any $k,h=1,\dots,n$, with $k\neq h$, we have $\mathrm{d}_{\boldsymbol{x}_o}f_k\neq\mathrm{d}_{\boldsymbol{x}_o}f_h$,*
- *$b_h-b_k\notin\mathbb{Z}^*$ whenever $f_h(\boldsymbol{x}_o)=f_k(\boldsymbol{x}_o)$.*

If $F_1,F_2\in\mathcal{A}$ are two formal power series solutions of the system $\mathrm{DE}_{d,n}((f_i)_{i=1}^n;(b_i)_{i=1}^n)$ such that $F_1(\boldsymbol{x}_o)=F_2(\boldsymbol{x}_o)$, then $F_1=F_2$.

In particular, given an initial condition $F_o\in M(n,\mathbb{C})$, with $F_o=F_o''$, there exists at most one holomorphic solution F of $\mathrm{DE}_{d,n}((f_i)_{i=1}^n;(b_i)_{i=1}^n)$ such that $F(x_o)=F_o$.

Proof. We have to show that the derivatives $\partial_{i_1}\dots\partial_{i_N}F_{kh}(\boldsymbol{x}_o)$ can be computed from the only knowledge of the numbers $F_{kh}(\boldsymbol{x}_o)$. We proceed by induction on N. Let us start with the case $N=1$.

Before proceeding with the proof, notice for any $k,h=1,\dots,n$, with $k\neq h$, there exists an index[11] $j_0\in\{1,\dots,d\}$ such that $\partial_{j_0}f_h(\boldsymbol{x}_o)\neq\partial_{j_0}f_k(\boldsymbol{x}_o)$, by assumption (1).

Step 1. Let $a\in\{1,\dots,d\}$ be such that $\partial_a f_h(\boldsymbol{x}_o)=\partial_a f_k(\boldsymbol{x}_o)$. Consider the equation (4.30) with specialization of indices $(i,j)=(a,j_0)$. By evaluation at $\boldsymbol{x}=\boldsymbol{x}_o$, we can compute the number $\partial_a F_{kh}(\boldsymbol{x}_o)$.

Step 2. Assume that $f_k(\boldsymbol{x}_o)\neq f_h(\boldsymbol{x}_o)$. Then, for any $a\in\{1,\dots,d\}$, we can compute all the number $\partial_a F_{kh}(\boldsymbol{x}_o)$ from equation (4.31), with specialzation of index $i=a$, by evaluation at $\boldsymbol{x}=\boldsymbol{x}_o$.

[11] Clearly j_0 depends on (k,h), but we omit the dependence for brevity of notation.

Step 3. Assume that $f_k(\boldsymbol{x}_o) = f_h(\boldsymbol{x}_o)$. Let $a \in \{1, \dots, d\}$ be such that $\partial_a f_k(\boldsymbol{x}_o) \neq \partial_a f_h(\boldsymbol{x}_o)$. Consider equation (4.31), with specialization of index $i = a$, and compute the ∂_{j_0}-derivative of both sides of the equation: we obtain

$$\begin{split}(\partial_{j_0} f_h - \partial_{j_0} f_k)\partial_a F_{kh} + (f_h - f_k)\partial^2_{aj_0} F_{kh} = \\ (b_h - b_k - 1)(\partial^2_{aj_0} f_h - \partial^2_{aj_0} f_k)F_{kh} + (b_h - b_k - 1)(\partial_a f_h - \partial_a f_k)\partial_{j_0} F_{kh} \\ + \partial_{j_0}\left[\sum_{\ell=1}^n (\partial_a f_\ell - \partial_a f_k)(f_h - f_\ell)F_{k\ell}F_{\ell h} - \sum_{\ell=1}^n (f_\ell - f_k)(\partial_a f_h - \partial_a f_\ell)F_{k\ell}F_{\ell h}\right],\end{split} \quad (4.32)$$

and the last summand can be further expanded by using Leibniz rule. By evaluating (4.32) at $\boldsymbol{x} = \boldsymbol{x}_o$, we obtain an identity of the type

$$\begin{split}(\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o))\partial_a F_{kh}(\boldsymbol{x}_o) \\ - (b_h - b_k - 1)(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o))\partial_{j_0} F_{kh}(\boldsymbol{x}_o) = X_1,\end{split} \quad (4.33)$$

where X_1 is an expression involving only

- the values at $\boldsymbol{x} = \boldsymbol{x}_o$ of f_h, f_k, and of their first and second partial derivatives,
- the values at $\boldsymbol{x} = \boldsymbol{x}_o$ of F, and of its first partial derivatives which can be computed in Step 2.

Similarly, consider equation (4.30), with specialization of indices $(i, j) = (a, j_0)$. By evaluation at $\boldsymbol{x} = \boldsymbol{x}_o$, we obtain an identity of the type

$$(\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o))\partial_a F_{kh}(\boldsymbol{x}_o) - (\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o))\partial_{j_0} F_{kh}(\boldsymbol{x}_o) = X_2, \quad (4.34)$$

where X_2 is an expression involving only

- the values at $\boldsymbol{x} = \boldsymbol{x}_o$ of first derivatives of f_h, f_k,
- the values at $\boldsymbol{x} = \boldsymbol{x}_o$ of F_{kh}.

Equations (4.33) and (4.34) define a linear system of equations

$$\begin{pmatrix} (\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o)) & -(b_h - b_k - 1)(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o)) \\ (\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o)) & -(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o)) \end{pmatrix}\begin{pmatrix} \partial_a F_{kh}(\boldsymbol{x}_o) \\ \partial_{j_0} F_{kh}(\boldsymbol{x}_o) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Such a system admits a unique solution since

$$\begin{split}\det\begin{pmatrix} (\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o)) & -(b_h - b_k - 1)(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o)) \\ (\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o)) & -(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o)) \end{pmatrix} = \\ = (\partial_{j_0} f_h(\boldsymbol{x}_o) - \partial_{j_0} f_k(\boldsymbol{x}_o))(\partial_a f_h(\boldsymbol{x}_o) - \partial_a f_k(\boldsymbol{x}_o))(b_h - b_k - 2) \neq 0.\end{split}$$

This proves that all the first derivatives $\partial_a F_{kh}(\boldsymbol{x}_o)$ can be computed.

Inductive step. Assume to know all the N-th derivatives $\partial_{i_1} \dots \partial_{i_N} F_{kh}(\boldsymbol{x}_o)$. We show how to compute all the $(N+1)$-th derivatives $\partial_{i_1} \dots \partial_{i_{N+1}} F_{kh}(\boldsymbol{x}_o)$.

Step 1. Assume there exists an $\ell \in \{1, \dots, N+1\}$ such that $\partial_{i_\ell} f_h(\boldsymbol{x}_o) = \partial_{i_\ell} f_k(\boldsymbol{x}_o)$. Without loss of generality, we can assume $\ell = N + 1$. Consider equation (4.30), with specialization of indices $(i, j) = (i_{N+1}, j_0)$, and take the $\partial_{i_1} \dots \partial_{i_N}$-derivative of both sides. By evaluation at $\boldsymbol{x} = \boldsymbol{x}_o$, we can compute the number $\partial_{i_1} \dots \partial_{i_{N+1}} F_{kh}(\boldsymbol{x}_o)$ in terms of lower order derivatives of F_{kh} at $\boldsymbol{x}_o$ (hence previously computed).

Step 2. Assume that $f_h(\boldsymbol{x}_o) \neq f_k(\boldsymbol{x}_o)$. Consider equation (4.31), with specialization of index $i = i_{N+1}$, and take the $\partial_{i_1} \dots \partial_{i_N}$-derivative of both sides. By evaluation at $\boldsymbol{x} = \boldsymbol{x}_o$, we can compute the number $\partial_{i_1} \dots \partial_{i_{N+1}} F_{kh}(\boldsymbol{x}_o)$ in terms of lower order derivatives of F_{kh} at $\boldsymbol{x}_o$ (hence previously computed).

Step 3. Assume that $f_h(\boldsymbol{x}_o) = f_k(\boldsymbol{x}_o)$, and that for any $\ell \in \{1, \dots, N+1\}$ we have $\partial_{i_\ell} f_h(\boldsymbol{x}_o) \neq \partial_{i_\ell} f_k(\boldsymbol{x}_o)$. Set

$$\partial_{\hat{0}} := \partial_{i_1}\partial_{i_2} \dots \partial_{i_{N+1}}, \quad \partial_{\hat{\ell}} := \partial_{j_0}\partial_{i_1} \dots \partial_{i_{\ell-1}}\partial_{i_{\ell+1}} \dots \partial_{i_{N+1}}, \quad \ell = 1, \dots, N+1.$$

For any $\ell \in \{1, \dots, N+1\}$, consider equation (4.30), with specialization of indices $(i, j) = (i_\ell, j_0)$. By taking the $\partial_{i_1} \dots \partial_{i_{\ell-1}}\partial_{i_{\ell+1}} \dots \partial_{i_{N+1}}$-derivative of both sides, we obtain an identity of the form

$$(\partial_{j_0} f_h - \partial_{j_0} f_k)\partial_{\hat{0}} F_{kh} - (\partial_{i_\ell} f_h - \partial_{i_\ell} f_k)\partial_{\hat{\ell}} F_{kh} = Z_{1,\ell}, \tag{4.35}$$

where $Z_{1,\ell}$ is a polynomial expression in the p-th derivatives of $(f_h - f_k)$, with $0 \leqslant p \leqslant N+1$, and the q-th derivatives of F with $0 \leqslant q \leqslant N$.

Consider equation (4.31), with specialization of index $i = i_{N+1}$. By taking the $\partial_{\widehat{N+1}}$-derivative of both sides we obtain an identity of the form

$$\begin{aligned}(f_h - f_k)\partial^{N+2}_{j_0 i_1 \dots i_{N+1}} F_{kh} + (\partial_{j_0} f_h - \partial_{j_0} f_k)\partial_{\hat{0}} F_{kh} + \sum_{\ell=1}^{N} (\partial_{i_\ell} f_h - \partial_{i_\ell} f_k)\partial_{\hat{\ell}} F_{kh} \\ - (b_h - b_k - 1)(\partial_{i_{N+1}} f_h - \partial_{i_{N+1}} f_k)\partial_{\widehat{N+1}} F_{kh} = Z_2, \end{aligned} \tag{4.36}$$

where Z_2 is a polynomial expression in

- the p-th derivatives of $f_h - f_k$, with $0 \leqslant p \leqslant N+2$,
- in the q-th derivatives of F, with $0 \leqslant q \leqslant N+1$.

Moreover, by evaluating at $\boldsymbol{x} = \boldsymbol{x}_o$ both sides of (4.36), one can notice that:

- the first term in the left-hand-side of (4.36) (i.e. the one with the $(N+2)$-th derivative of F_{kh}) cancels,
- the only $(N+1)$-th derivatives of F appearing in $Z_2(\boldsymbol{x}_o)$ are those computed in Step 2.

Hence, equations (4.35) evaluated at $\boldsymbol{x} = \boldsymbol{x}_o$, for $\ell = 1, \dots, N+1$, and equations (4.36) evaluated at $\boldsymbol{x} = \boldsymbol{x}_o$, define a linear system of equations in the $N+2$ unknowns $\partial_{\hat{0}} F_{kh}(\boldsymbol{x}_o)$, $\partial_{\hat{1}} F_{kh}(\boldsymbol{x}_o)$, $\dots$, $\partial_{\widehat{N+1}} F_{kh}(\boldsymbol{x}_o)$:

$$W \begin{pmatrix} \partial_{\hat{0}} F_{kh}(\boldsymbol{x}_o) \\ \partial_{\hat{1}} F_{kh}(\boldsymbol{x}_o) \\ \vdots \\ \partial_{\widehat{N}} F_{kh}(\boldsymbol{x}_o) \\ \partial_{\widehat{N+1}} F_{kh}(\boldsymbol{x}_o) \end{pmatrix} = \begin{pmatrix} Z_{1,1}(\boldsymbol{x}_o) \\ Z_{1,2}(\boldsymbol{x}_o) \\ \vdots \\ Z_{1,N+1}(\boldsymbol{x}_o) \\ Z_2(\boldsymbol{x}_o) \end{pmatrix},$$

where the matrix W equals

$$W=\begin{pmatrix} D_0 & -D_1 & 0 & 0 & \dots & 0 & 0\\ D_0 & 0 & -D_2 & 0 & \dots & 0 & 0\\ D_0 & 0 & 0 & -D_3 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ D_0 & 0 & 0 & 0 & \dots & -D_N & 0\\ D_0 & 0 & 0 & 0 & \dots & 0 & -D_{N+1}\\ D_0 & D_1 & D_2 & D_3 & \dots & D_N & -\kappa D_{N+1}\end{pmatrix},\qquad \begin{gathered} D_0:=\partial_{j_0}f_h(\boldsymbol{x}_o)-\partial_{j_0}f_k(\boldsymbol{x}_o),\\ D_\ell:=\partial_{i_\ell}f_h(\boldsymbol{x}_o)-\partial_{i_\ell}f_k(\boldsymbol{x}_o),\\ \kappa=b_h-b_k-1.\end{gathered}$$

We have

$$\det W=(\partial_{j_0}f_h(\boldsymbol{x}_o)-\partial_{j_0}f_k(\boldsymbol{x}_o))\prod_{\ell=1}^{N+1}(\partial_{i_\ell}f_h(\boldsymbol{x}_o)-\partial_{i_\ell}f_k(\boldsymbol{x}_o))(N+1-b_h+b_k)\neq 0.$$

This proves that all the $(N+1)$-th derivatives $\partial_{i_1}\dots\partial_{i_{N+1}}F_{kh}(\boldsymbol{x}_o)$ can be computed. $\square$

4.7. **$\mathfrak{J}$-universal integrable deformations.** Consider a meromorphic connection ∇^o on a vector bundle $E^o\to\mathbb{P}^1$, with matrix of connection 1-forms

$$\Omega_o=-\left(A_o+\frac{1}{z}B_o\right)\mathrm{d}z.$$

Assume that one of the following assumptions holds true:

(I) $A_o\in\mathcal{M}_{\rm diag}\cap\mathcal{M}_{\rm reg}$;
(II) $A_o\in\mathcal{M}_{\rm diag}$, and the Property PNR is satisfied.

Definition 4.29. Let $\mathfrak{J}$ be a class of integrable deformations of ∇^o. An integrable deformation (∇,E,X,x_o) of ∇^o is $\mathfrak{J}$-*versal*, if

- (∇,E,X,x_o) is an element of $\mathfrak{J}$,
- any element (∇',E',X',x_o') of $\mathfrak{J}$ is induced by (∇,E,X,x_o) via pull-back along a base-change $\varphi\colon(X',x_o')\to(X,x_o)$.

It is $\mathfrak{J}$-*universal*, if the germ at x_o' of the base change φ is uniquely determined.

Remark 4.30. Given a integrable deformation ∇ of the connection ∇^o, there always exist classes $\mathfrak{J}$ such that ∇ is $\mathfrak{J}$-versal. For example, ∇ is clearly $\{\nabla\}$-versal. This is not true for universality, due to possible "internal symmetries" of the integrable deformations. Here is an example. Let $f\colon\mathbb{C}\to\mathbb{C}$ be an arbitrary holomorphic function, and $d_1,d_2\in\mathbb{C}$. Consider the matrices $\Delta_0,\mathfrak{B}\colon\mathbb{C}^2\to M(2,\mathbb{C})$ and the 1-form $\omega\colon\mathbb{C}^2\to M(2,\mathbb{C})\otimes\Omega^1_{\mathbb{C}}$ defined by

$$\Delta_0(x)=\begin{pmatrix} f(x_1^2+x_2^2) & 0\\ 0 & f(x_1^2+x_2^2)\end{pmatrix},\quad \mathfrak{B}(x)=\begin{pmatrix} d_1 & (d_1-d_2)(x_1^2+x_2^2)\\ 0 & d_2\end{pmatrix},$$

$$\omega(x)=\begin{pmatrix}0&2\\0&0\end{pmatrix}(x_1\mathrm{d}x_1+x_2\mathrm{d}x_2).$$

The matrices

$$\widetilde{\Omega}_o(z)=-\left(\Delta_0(0)+\frac{1}{z}\mathfrak{B}(0)\right)\mathrm{d}z,\quad \widetilde{\Omega}(z,x)=-\left(\Delta_0(x)+\frac{1}{z}\mathfrak{B}(x)\right)\mathrm{d}z-z\,\mathrm{d}\Delta_0(x)+\omega,$$

define two connections ∇^o and ∇, respectively. The connection ∇ is an integrable deformation of ∇^o, at which it restricts at $x=0$. The connection ∇ is not $\{\nabla\}$-universal: any linear map $l\colon \mathbb{C}^2\to\mathbb{C}^2$, defined by a matrix in $O(2,\mathbb{C})$, is such that $l^*\widetilde{\Omega}=\widetilde{\Omega}$.

Let us denote[12] by $(\nabla^{\rm JMUMS},\underline{\mathbb{C}^n},\mathbb{D},\boldsymbol{u}_o)$ the integrable deformation of ∇^o whose existence is guaranteed by Theorems 4.8 and 4.10 (Case I), and Theorem 4.11 (Case II). Here $\mathbb{D}\subseteq\mathbb{C}^n$ denotes a sufficiently small polydisc centered at $\boldsymbol{u}_o$. Recall that $\nabla^{\rm JMUMS}$ has matrix of connection 1-forms given by

$$\widehat{\Omega}_{\rm JMUMS}(z,\boldsymbol{u})=-{\rm d}\left(z\Lambda(\boldsymbol{u})\right)-\left([\Gamma(\boldsymbol{u}),\Lambda(\boldsymbol{u})]+\mathcal{B}'_o\right)\frac{{\rm d}z}{z}-[\Gamma(\boldsymbol{u}),{\rm d}\Lambda(\boldsymbol{u})],$$

where $\Lambda\colon\mathbb{C}^n\to M(n,\mathbb{C})$, is defined by $\Lambda(\boldsymbol{u})={\rm diag}(u^1,\dots,u^n)$, and where the matrix $\mathcal{B}'_o={\rm diag}(b_1,\dots,b_n)$ is constant, and satisfying Property PNR-($\star\star$) in Case II.

Theorem 4.31. *Let $\mathfrak{I}$ be a class of integrable deformations of the connection ∇^o satisfying conditions (I) or (II) above.*

1. *An integrable deformation of ∇^o is induced by $(\nabla^{\rm JMUMS},\underline{\mathbb{C}^n},\mathbb{D},\boldsymbol{u}_o)$ only if it is of fs-type.*
2. *If the deformation $(\nabla^{\rm JMUMS},\underline{\mathbb{C}^n},\mathbb{D},\boldsymbol{u}_o)$ is $\mathfrak{I}$-versal, then it is $\mathfrak{I}$-universal.*
3. *There exists a unique maximal class $\mathfrak{I}_{\rm JMUMS}$ of integrable deformations of ∇^o such that $\nabla^{\rm JMUMS}$ is $\mathfrak{I}_{\rm JMUMS}$-universal.*
4. *In Case I, we have $\mathfrak{I}_{\rm JMUMS}=\mathfrak{I}(\nabla^o)$.*
5. *In Case II, we have $\mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o)\subseteq\mathfrak{I}_{\rm JMUMS}\subseteq\mathfrak{I}_{\rm fs}(\nabla^o)$.*

Proof. Point (1) follows from point (1) of Theorem 4.24, since $\nabla^{\rm JMUMS}\in\mathfrak{I}_{\rm fs}(\nabla^o)$.

Assume that $(\nabla,E,X,x_o)\in\mathfrak{I}_{\rm fs}(\nabla^o)$ is induced by $(\nabla^{\rm JMUMS},\underline{\mathbb{C}^n},\mathbb{D},\boldsymbol{u}_o)$ via a holomorphic map $\varphi\colon(X,x_o)\to(\mathbb{C}^n,\boldsymbol{u}_o)$, $x\mapsto(\varphi_1(x),\dots,\varphi_n(x))$. On the one hand, by Theorems 4.16, 4.19, and 4.24, the pole part of ∇ is locally holomorphically diagonalizable at x_o, with holomorphic diagonal form $\Delta_0(x)=(f_1(x),\dots,f_n(x))$. On the other hand, the connection ∇ has then matrix of connection 1-forms $\varphi^*\widehat{\Omega}_{\rm JMUMS}$: in particular, its pole part has Jordan diagonal form

$$(\varphi^*\Lambda)(x)={\rm diag}(\varphi^1(x),\dots,\varphi^n(x)),\quad x\in X.$$

Hence, the germ of the map φ at x_o is uniquely determined: it is given by the *spectrum map* $\sigma\colon X\to\mathbb{C}^n$, $x\mapsto(f_1(x),\dots,f_n(x))$. This proves point (2).

The class $\mathfrak{A}=\{\mathfrak{I}\subseteq\mathfrak{I}(\nabla^o)\colon \nabla^{\rm JMUMS}\text{ is }\mathfrak{I}\text{-universal}\}$ is non-empty, since $\{\nabla^{\rm JMUMS}\}\in\mathfrak{A}$. Consider the poset $(\mathfrak{A},\subseteq)$. By taking unions, it is easy to see that

(i) every chain in $\mathfrak{A}$ has a maximal element,
(ii) and $\mathfrak{A}$ is upward-directed (i.e. given $\mathfrak{I}_1,\mathfrak{I}_2\in\mathfrak{A}$, there exists $\mathfrak{I}_3\in\mathfrak{A}$ such that $\mathfrak{I}_1,\mathfrak{I}_2\subseteq\mathfrak{I}_3$).

By Zorn Lemma, $\mathfrak{A}$ has a maximal element. It necessarily is unique, by (ii). This proves point (3).

In Case I, the statement $\mathfrak{I}_{\rm JMUMS}=\mathfrak{I}(\nabla^o)$ is equivalent to the universality predicated in Theorems 4.8 and 4.10. Hence (4) holds.

[12]Here the superscript "JMUMS" stands for Jimbo–Miwa–Ueno–Malgrange–Sabbah.

The only non-tivial inculsion of point (5) is $\mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o)\subseteq\mathfrak{I}^{\rm JMUMS}$. Consider an integrable deformation $(\nabla,E,X,x_o)\in\mathfrak{I}^{\rm gen}_{\rm d}(\nabla^o)$. By Theorem 4.24, ∇ is of dv-type, and it has a matrix of connection 1-forms

$$\widehat{\Omega}(z,x)=-\left(\Delta_0(x)+\frac{1}{z}(\mathcal{B}'_o+[L(x),\Delta_0])\right){\rm d}z-z{\rm d}\Delta_0(x)-[L(x),{\rm d}\Delta_0(x)],$$

where $\mathcal{B}'_o$ satisfies Property PNR-($\star\star$), and (Δ_0,L) is a suitable pair of holomorphic matrices defined in a neighborhood of x_o. Consider spectrum map $\sigma\colon(X,x_o)\to(\mathbb{C}^n,\boldsymbol{u}_o)$ defined by $x\mapsto(f_1(x),\dots,f_n(x))$. Since $\sigma^*\Lambda=\Delta_0$, we have

$$\widehat{\Omega}-\sigma^*\widehat{\Omega}_{\rm JMUMS}=-[L-\sigma^*\Gamma,\Delta_0]\frac{{\rm d}z}{z}-[L-\sigma^*\Gamma,{\rm d}\Delta_0].$$

Moreover, since the restriction at x_o of both $\widehat{\Omega}$ and $\sigma^*\widehat{\Omega}_{\rm JMUMS}$ equals (4.6), we deduce that

$$L(x_o)-(\sigma^*\Gamma)(x_o)\in\mathcal{K}\,[[-,\Delta_0];x_o]\equiv\ker[-,{\rm d}\Delta_0(x_o)],\qquad\text{that is }L(x_o)=(\sigma^*\Gamma)(x_o).$$

Since both L and $\sigma^*\Gamma$ solves the generalized Darboux–Egoroff equations (4.17), (4.18), we conclude that $L=\sigma^*\Gamma$, by Theorem 4.28. This shows that $\nabla=\sigma^*\nabla^{\rm JMUMS}$. This completes the proof. ☐

Remark 4.32. From the equality $L=\sigma^*\Gamma$, we are able to justify the estimate of Remark 4.26. Consider equation (4.12): by evaluating both sides at $\boldsymbol{u}(x)=(f_1(x),\dots,f_n(x))$, we obtain

$$\begin{aligned}(b_j-b_i-1)L_{ij}(x)-\sum_{\ell\neq i}(f_\ell(x)-f_i(x))L_{i\ell}(x)L_{\ell j}(x)&=(f_i(x)-f_j(x))\cdot\partial_j\Gamma_{ij}|_{\boldsymbol{u}(x)}\\&=O\left(f_i(x)-f_j(x)\right),\end{aligned}\tag{4.37}$$

as expected from the proof of Theorem 4.25.

The following result provides a further description of the class $\mathfrak{I}_{\rm JMUMS}$.

Proposition 4.33. *Let ∇^o satisfy condition (I) or (II). Consider an integrable deformation (∇,E,X,x_o) in $\mathfrak{I}_{\rm fs}(\nabla^o)$, with pole part $\Delta_0(x)=(f_1(x),\dots,f_n(x))$.*

1. *The connection ∇ is formally gauge equivalent to the connection $\sigma^*\nabla^{\rm JMUMS}$, where $\sigma\colon X\to\mathbb{C}^n$ is the spectrum map $x\mapsto(f_1(x),\dots,f_n(x))$. This means that there exist holomorphic functions $\Phi_k\colon X\to M(n,\mathbb{C})$, with $\Phi_0(X)\subseteq GL(n,\mathbb{C})$, such that*
$$\Phi^{-1}\widehat{\Omega}\Phi+\Phi^{-1}{\rm d}\Phi=\sigma^*\widehat{\Omega}_{\rm JMUMS},\qquad\Phi(z,x)=\sum_{k\geqslant0}\Phi_k(x)z^{-k}.$$
Moreover, if $\Phi_0(x)={\rm Id}_n$ for any $x\in X$, such a formal gauge equivalence is unique.
2. *We have $\nabla\in\mathfrak{I}_{\rm JMUMS}$ if and only if the formal gauge equivalence above is actually convergent.*

Proof. Both ∇ and $\sigma^*\nabla^{\rm JMUMS}$ are formally simplifiable: they are formally equivalent to the connection ${\rm d}-{\rm d}\left(z\Delta_0(x)\right)-\mathfrak{B}'_o\frac{{\rm d}z}{z}$, via unique formal gauge equivalences of the form ${\rm Id}_n+O(z^{-1})$. Point (1) follows.

If $\nabla\in\mathfrak{I}_{\rm JMUMS}$, then there exists an analytic gauge equivalence of the form $T(z,x)={\rm Id}_n+\sum_{k\geqslant1}T_k(x)z^{-k}$ such that $T^{-1}\widehat{\Omega}T+T^{-1}{\rm d}T=\sigma^*\widehat{\Omega}_{\rm JMUMS}$. By uniqueness, we have $T=\Phi$. Conversely, if Φ is convergent, then ∇ and $\sigma^*\nabla^{\rm JMUMS}$ are analytically gauge equivalent. ☐

Appendix A.

A.1. **Proof of Theorem 4.7.** Consider a trivial vector bundle E^o on $\mathbb{P}^1$, equipped with a meromoprhic connection ∇^o, admitting (in a suitable basis of sections) a matrix of 1-forms connection of the form (4.5) with A_o regular. Let $(E, \nabla, \boldsymbol{x}_o)$ be an arbitrary integrable deformation of (E^o, ∇^o), parametrized by a simply connected manifold X. By Theorem 4.5, the deformation admits a matrix of connection 1-forms as in (4.2), whose coefficients are subjected to the integrability equations (4.3).

Since A_o is regular, the matrix $A(\boldsymbol{x})$ is regular for $\boldsymbol{x}$ is a sufficiently small neighborhood $U \subseteq X$ of $\boldsymbol{x}_o$. From the equation $[A, C] = 0$, we deduce that $C(\boldsymbol{x})$ is a polynomial expression of $A(\boldsymbol{x})$ for $\boldsymbol{x} \in U$. Hence, the equations $C \wedge C = 0$ are automatically satisfied for $\boldsymbol{x} \in U$.

Since $\mathrm{d}C = 0$, at least locally we have $C = \mathrm{d}K$ for some holomorphic matrix-valued function K: up to adding a constant matrix, we can assume that $K(x_o) = 0$. Consequently, system (4.3) is equivalent to

$$A - K - [K, B_o] = \text{const.} = A_o, \qquad\qquad [A, \mathrm{d}K] = 0.$$

These equations define a Pfaffian system for the matrix K only:

$$\omega = [A_o + K + [K, B_o], \mathrm{d}K] = 0. \tag{A.1}$$

Given a solution K, the matrix A can be reconstructed by $A = A_o + K + [K, B_o]$.

The argument above shows that there is a 1-1 correspondence between integrable deformations of a connection (E^o, ∇^o) and germs of maps $\varphi\colon (X, x_o) \to (M(n, \mathbb{C}), 0)$ such that $\varphi^*\omega = 0$. The maps φ are the integral manifolds of the Pfaffian system (A.1) passing through $0 \in M(n, \mathbb{C})$.

Lemma A.1. *If A_o is regular, then the Pfaffian system (A.1) is completely integrable on $M(n, \mathbb{C})$. The maximal integral solutions define a foliation: in a neighborhood of $0 \in M(n, \mathbb{C})$, the leaves are n-dimensional.*

Proof. Let v_1, v_2 be two vector fields on $M(n, \mathbb{C})$ defined in a neighborhood of the origin. We have to show that if $\omega(v_i) = 0$ for $i = 1, 2$, then also $\mathrm{d}\omega(v_1 \wedge v_2) = 0$. A simple computation shows that $\mathrm{d}\omega = 2\mathrm{d}K \wedge \mathrm{d}K + 2[\mathrm{d}K \wedge \mathrm{d}K, B_o]$. By identifying v_1, v_2 with matrices in $M(n, \mathbb{C})$, the condition $\omega(v_i) = 0$ is equivalent to $[A(x), v_i] = 0$, for $i = 1, 2$. For $x \in U$ as above, the matrix $A(x)$ is regular, and the matrices v_1, v_2 are polynomials in $A(x)$. In particular, we have $[v_1, v_2] = 0$, so that $2[v_1, v_2] + 2[[v_1, v_2], B_o] = 0$. This is exactly the condition $\mathrm{d}\omega(v_1, v_2) = 0$. Finally, notice that the dimension of the leaf passing through 0 equals the dimension of the centralizer of A_o in $M(n, \mathbb{C})$. This equals n, since A_o is regular. □

The germ of the universal deformation of (E^o, ∇^o) is given by the germ of the maximal integral submanifold of the Pfaffian system (A.1) passing through 0. This completes the proof of Theorem 4.7.

A.2. **Versal deformations do not exist if $A_o \notin \mathcal{M}_{\mathrm{reg}}$.** Let $n = 2$, and introduce the matrices

$$A_o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_o = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}. \tag{A.2}$$

Let ∇^o be the connection on $\underline{\mathbb{C}^2} \to \mathbb{P}^1$ with connection (4.5). Consider an integrable deformation of ∇^o defined by a matrix $\Omega(z, \boldsymbol{x})$ as in (4.2), where $\boldsymbol{x}$ is a parameter varying in a polydisc $\mathbb{D} \subseteq \mathbb{C}^m$. The matrix

$$\Omega'(z, \boldsymbol{x}, s) := \Omega(z, \boldsymbol{x}) + \mathrm{d}(zs\,\mathrm{Id}_2)$$

defines a new integrable deformation of ∇^o, parametrized by points $(\boldsymbol{x}, s) \in \mathbb{D}\times\mathbb{C}$. If $A(\boldsymbol{x})$ is the pole part of Ω, consider the hypersurface $\mathcal{L} := \{s = \frac{1}{2}\mathrm{Tr}\,A(\boldsymbol{x})\}$ in $\mathbb{D}\times\mathbb{C}$: the restriction $\Omega'|_{\mathbb{P}^1\times\mathcal{L}}$ is a deformation of ∇^o with traceless pole part. This shows that, without loss of generality, we may restrict to the study of integrable deformations with traceless pole part

$$A(\boldsymbol{x}) = \begin{pmatrix} \alpha(\boldsymbol{x}) & \beta(\boldsymbol{x}) \\ \gamma(\boldsymbol{x}) & -\alpha(\boldsymbol{x}) \end{pmatrix}.$$

As in the previous section, the integrability system (4.3) can be reduced to the following system of equations in the pair (A, K), with $\mathrm{d}K = C$:

$$[A, \mathrm{d}K] = 0, \quad A = K + [K, B_o], \quad \mathrm{d}K \wedge \mathrm{d}K = 0. \tag{A.3}$$

Assume that $c \neq \pm 1$, so that the operator

$$\nu\colon M(2,\mathbb{C}) \to M(2,\mathbb{C}), \quad X \mapsto X + [X, B_o],$$

is invertible, with inverse

$$\nu^{-1}\colon \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & \frac{1}{1-c}x_{12} \\ \frac{1}{1+c}x_{21} & x_{22} \end{pmatrix}.$$

The system (A.3) can be reduced to a system of differential equations in A only, since $K = \nu^{-1}(A)$. The equation $[A, \mathrm{d}K] = 0$ becomes

$$\begin{pmatrix} 0 & -2\beta \\ 2\gamma & 0 \end{pmatrix} \mathrm{d}\alpha + \begin{pmatrix} -\gamma & 2\alpha \\ 0 & \gamma \end{pmatrix} \frac{\mathrm{d}\beta}{1-c} + \begin{pmatrix} \beta & 0 \\ -2\alpha & -\beta \end{pmatrix} \frac{\mathrm{d}\gamma}{1+c} = 0,$$

which defines following the Pfaffian system on the space $\mathbb{C}^3$ of triples (α, β, γ):

$$\begin{aligned} \omega_1 &= (1-c)\beta\,\mathrm{d}\alpha - \alpha\,\mathrm{d}\beta = 0, \\ \omega_2 &= (1+c)\gamma\,\mathrm{d}\alpha - \alpha\,\mathrm{d}\gamma = 0, \\ \omega_3 &= (1+c)\gamma\,\mathrm{d}\beta - (1-c)\beta\,\mathrm{d}\gamma = 0. \end{aligned} \tag{A.4}$$

This system can be written in a more compact way as

$$\frac{\mathrm{d}\alpha}{\alpha} = \frac{\mathrm{d}\beta}{(1-c)\beta} = \frac{\mathrm{d}\gamma}{(1+c)\gamma}.$$

The equation $\mathrm{d}K \wedge \mathrm{d}K = 0$ reduces to

$$\mathrm{d}\alpha \wedge \mathrm{d}\beta = \mathrm{d}\alpha \wedge \mathrm{d}\gamma = \mathrm{d}\beta \wedge \mathrm{d}\gamma = 0,$$

which are automatically satisfied if (A.4) holds true.

The discussion in the previous section shows that integrable deformations of ∇^o, with traceless pole part, define (and are defined by) germs of maps $\varphi\colon (X, \boldsymbol{x}_o) \to (\mathbb{C}^3, A_o)$ such that $\varphi^*\omega_i = 0$ for $i = 1, 2, 3$, i.e. integral submanifolds of the Pfaffian system (A.4) passing through $(\alpha_o, \beta_o, \gamma_o) = 0$.

If $(\alpha_o, \beta_o, \gamma_o) \neq 0$, there exists a unique (one dimensional) maximal integral submanifold of the Pfaffian system above passing through $(\alpha_o, \beta_o, \gamma_o)$:

$$\alpha(t) = \alpha_o e^t, \qquad \beta(t) = \beta_o e^{(1-c)t}, \qquad \gamma(t) = \gamma_o e^{(1+c)t}.$$

If $(\alpha_o, \beta_o, \gamma_o) = 0$, on the other hand, one can find many integral curves passing through $(\alpha_o, \beta_o, \gamma_o)$. For example:

$$\alpha = t, \quad \beta = \gamma = 0; \qquad \alpha = \gamma = 0, \quad \beta = t; \qquad \alpha = \beta = 0, \quad \gamma = t;$$

these lines are not contained in a surface, hence there is no versal deformation inducing all of them.

Infinite families of solutions arise if $c \in \mathbb{Q}$. Assume $c = \frac{p}{q}$ with $(p, q) = 1$, and $q > 0$:

- if $c < -1$ (i.e. $p < -q$) we have the infinite family
$$\alpha = \alpha_0 t^q, \qquad \beta = \beta_0 t^{q-p}, \qquad \gamma = 0, \quad \alpha_0, \beta_0 \in \mathbb{C};$$
- if $-1 < c < 1$ (i.e. $-q < p < q$) we also have another infinite family
$$\alpha = \alpha_0 t^q, \qquad \beta = \beta_0 t^{q-p}, \qquad \gamma = \gamma_0 t^{p+q}, \quad \alpha_0, \beta_0, \gamma_0 \in \mathbb{C};$$
if moreover $p + q$ is even, then we also have the infinite family
$$\alpha = 0, \qquad \beta = \beta_0 t^{\frac{q-p}{2}}, \qquad \gamma = \gamma_0 t^{\frac{p+q}{2}}, \quad \beta_0, \gamma_0 \in \mathbb{C};$$
- if $c > 1$ (i.e. $p > q$) we have the infinite family
$$\alpha = \alpha_0 t^q, \qquad \beta = 0, \qquad \gamma = \gamma_0 t^{p+q}, \quad \alpha_0, \gamma_0 \in \mathbb{C}.$$

For a complete classification of the solutions see the recent paper [Her21, Sec. 8].

A.3. **Case of $\mathcal{B}'_o$ partially resonant.** Let us now consider the case of the matrices A_o, B_o as in (A.2) with $c = 1$. For such a pair of matrices the Property PNR cannot hold true. As before, let ∇^o be the connection on $\underline{\mathbb{C}^2} \to \mathbb{P}^1$ with connection (4.5), and consider an arbitrary (germ of) integrable deformation ∇ of ∇^o, parametrized by (the germ of) a pointed manifold $(X, \boldsymbol{x}_o)$. Let the matrix Ω of 1-forms of ∇ to be

$$\Omega(z, \boldsymbol{x}) = -\left(A(\boldsymbol{x}) + \frac{1}{z} B_o\right) \mathrm{d}z - zC(\boldsymbol{x}),$$

as in Theorem 4.5.

Proposition A.2. *The matrix Ω is of one, and only one, of the following types:*
Type I: *There exist two holomorphic functions $g, m\colon X \to \mathbb{C}$, not identically equal, with $g(\boldsymbol{x}_o) = m(\boldsymbol{x}_o) = 0$, and a complex* modulus *$\kappa \in \mathbb{C}$ such that*

$$A = \begin{pmatrix} g & 0 \\ 2\kappa(m-g)^2 & m \end{pmatrix}, \qquad C = \begin{pmatrix} \mathrm{d}g & 0 \\ \kappa \cdot \mathrm{d}(m-g)^2 & \mathrm{d}m \end{pmatrix}. \tag{A.5}$$

Type II: *There exist a holomorphic function $g\colon X \to \mathbb{C}$, with $g(\boldsymbol{x}_o) = 0$, such that*

$$A = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \qquad C = \begin{pmatrix} \mathrm{d}g & 0 \\ 0 & \mathrm{d}g \end{pmatrix}. \tag{A.6}$$

Type III: *There exist two holomorphic functions $g, h\colon X \to \mathbb{C}$, with h not identically zero, and $g(\boldsymbol{x}_o) = h(\boldsymbol{x}_o) = 0$, such that*

$$A(\boldsymbol{x}) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \qquad C(\boldsymbol{x}) = \begin{pmatrix} \mathrm{d}g & \mathrm{d}h \\ 0 & \mathrm{d}g \end{pmatrix}. \tag{A.7}$$

Proof. The integrability conditions (4.3) can be put in the form (A.3), but we cannot reduce anymore the problem to a Pfaffian system for $A(\boldsymbol{x})$, as we did in the previous section, since $c = 1$. So, let

$$K(\boldsymbol{x}) = \begin{pmatrix} g(\boldsymbol{x}) & h(\boldsymbol{x}) \\ \ell(\boldsymbol{x}) & m(\boldsymbol{x}) \end{pmatrix},$$

for arbitrary functions $g, h, \ell, m\colon X \to \mathbb{C}$ all vanishing at $\boldsymbol{x}_o$, and let us consider the system (A.3) in the variable K. The first equation of the system (A.3) is then equivalent to the set of equations

$$\ell\,\mathrm{d}h = 0, \qquad (g-m)\mathrm{d}h = 0, \qquad (m-g)\mathrm{d}\ell - 2\ell\,\mathrm{d}(m-g) = 0. \tag{A.8}$$

We may have two cases: (Case I) either h is identically zero on X, (Case II) or $h(\boldsymbol{x}) \neq 0$ for at least one $\boldsymbol{x} \in X$.

In Case I, from (A.8) we deduce that $\ell = \kappa(m-g)^2$ for some $\kappa \in \mathbb{C}$, and the remaining equation $\mathrm{d}K \wedge \mathrm{d}K = 0$ is automatically satisfied. Consequently, the matrix of connection 1-forms is necessarily of the form (A.5) or (A.6). In case II, from (A.8) we deduce that $\ell = 0$ and $g = m$, and the remaining equation $\mathrm{d}K \wedge \mathrm{d}K = 0$ is automatically satisfied. Hence, the matrix Ω of connection 1-forms is of the form (A.7). $\square$

Proposition A.3.

(1) *All integrable deformations in $\mathfrak{I}_{\mathrm{d}}(\nabla^o)$ are of Type I or II. Moreover, any element of $\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)$ is of Type I.*
(2) *Integrable deformations of Type I cannot induce deformations of Type III, and vice-versa.*
(3) *Both deformations of Type I and III can induce deformations of Type II.*
(4) *Two deformations $(\nabla^{[1]}, \kappa_1), (\nabla^{[2]}, \kappa_2)$ of Type I cannot be induced by a same deformation if $\kappa_1 \neq \kappa_2$.*
(5) *Integrable deformations of Type I and fixed modulus κ are induced by a universal deformation of such a type: it is the connection on $\underline{\mathbb{C}^2} \to \mathbb{P}^1 \times \mathbb{C}^2$ with matrix of connection 1-forms*

$$\Omega^{[\mathrm{I}]}_{\kappa}(z, \boldsymbol{u}) = -\mathrm{d}\left(z\Lambda(\boldsymbol{u})\right) - \left([F_\kappa, \Lambda(\boldsymbol{u})] + B_o\right)\frac{\mathrm{d}z}{z} - [F_\kappa, \mathrm{d}\Lambda(\boldsymbol{u})],$$

where

$$\Lambda(\boldsymbol{u}) = \mathrm{diag}(u_1, u_2), \quad F_\kappa = \begin{pmatrix} 0 & 0 \\ -2\kappa & 0 \end{pmatrix}.$$

(6) *Integrable deformations of Type II do not admit a versal deformation.*
(7) *Integrable deformations of Type III admit a versal (but not universal) deformation of such a type: it is the connection on $\underline{\mathbb{C}^2} \to \mathbb{P}^1 \times \mathbb{C}^2$ with matrix of connection 1-forms*

$$\Omega^{[\mathrm{III}]}(z, \boldsymbol{u}) = -\left(\begin{pmatrix} u_1 & 0 \\ 0 & u_1 \end{pmatrix} + \frac{1}{z}B_o\right)\mathrm{d}z - z\begin{pmatrix} \mathrm{d}u_1 & \mathrm{d}u_2 \\ 0 & \mathrm{d}u_1 \end{pmatrix}.$$

Proof. In all Types I, II, and III, the matrix $A(\boldsymbol{x})$ is holomorphically diagonalizable: for Type I, indeed, we have

$$M^{-1}AM = \mathrm{diag}(g(\boldsymbol{x}), m(\boldsymbol{x})), \qquad M(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 2\kappa(g(\boldsymbol{x}) - m(\boldsymbol{x})) & 1 \end{pmatrix};$$

in Type II and III, the matrix A is already in diagonal form. However, the deformation part C is diagonalizable only in Types I and II. In Type I we have $M^{-1}CM = \mathrm{diag}(\mathrm{d}g, \mathrm{d}m)$. Moreover, any element of $\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)$ necessarily is of Type I. This proves point (1).

All the remaining statements easily follow from the explicit equations (A.5), (A.6), and (A.7). $\square$

Points (1) and (4) of Proposition A.3 imply that there exist no $\mathfrak{I}^{\mathrm{gen}}_{\mathrm{d}}(\nabla^o)$-versal integrable deformations.

References

[Arn71] V. I. Arnol'd, *On matrices depending on parameters*, Uspekhi Mat. Nauk, 26:2(158) (1971), 101–114; Russian Math. Surveys, 26:2 (1971), 29–43.

[Bal89] W. Balser, *Meromorphic transformation to Birkhoff standard form in dimension three*, J. Fac. Sc. Univ. Tokyo 36 (1989), 233–246.

[Bal90] W. Balser, *Analytic transformation to Birkhoff standard form in dimension three*, Funk. Ekvac. 33 (1990), 59–67.

[Bau74] H. Baumgärtel, *Analytic perturbation theory for linear operators depending on several complex variables*, Mat. Issled. 9, 1, 17–39 (1974) (Russian).

[Bau85] H. Baumgärtel, *Analytic perturbation theory for matrices and operators*, Birkhäuser, 1985.

[BB97] W. Balser, A.A. Bolibrukh, *Transformation of reducible equations to Birkhoff standard form*, Ulmer Seminare (1997) 73–81.

[Bel16] P. Belmans, *Segre symbols*, unpublished note available at the webpage: https://pbelmans.ncag.info/notes/segre.pdf.

[BHM98] R. Byers, C. He, and V. Mehrmann, *Where is the nearest non-regular pencil?*, Lin. Alg. Appl., 121 (1998), pp. 245–287.

[Bir09] G. D. Birkhoff, *Singular points of ordinary linear differential equations*, Trans. Amer. Math. Soc. 10 (1909), p. 436–470.

[Bir13] G.D. Birkhoff, *A theorem on matrices of analytic functions*, Math. Ann. 74 (1913), 122–133.

[BJL79] W. Balser, W.B. Jurkat, and D.A. Lutz, *Birkhoff invariants and Stokes multipliers for meromorphic linear differential equations*, J. Math. Anal. Appl. 71 (1979), 48–94.

[BKL75] H. Bart, M.A. Kaashoek, D.C. Lay, *Relative Inverses of Meromorphic Operator Functions and Associated Holomorphic Projection Function*, Math. Ann. 218 (1975): 199–210.

[Bol94a] A.A. Bolibruch, *On analytic transformation to Birkhoff standard form*, Proc. Steklov Inst. Math. 203 (1994), 29–35.

[Bol94b] A.A. Bolibruch, *On analytic transformation to Birkhoff standard form*, Russian Acad. Sci. Dokl. Math. 49 (1994), 150–153.

[Cay55] A. Cayley, *Recherches sur les matrices dont les termes sont des fonctions linéaires d'une seule indéterminée*, J. Reine Angew. Math. 50 (1855), 313–317.

[CDG19] G. Cotti, B. Dubrovin, and D. Guzzetti, *Isomonodromy deformations at an irregular singularity with coalescing eigenvalues*, Duke Math. J. 168 (2019), no. 6, 967–1108. doi:10.1215/00127094-2018-0059.

[CDG20] G. Cotti, B. Dubrovin, and D. Guzzetti, *Local moduli of semisimple Frobenius coalescent structures*, Symmetry, Integrability and Geometry: Methods and Applications, 16: 040 (2020), doi:10.3842/SIGMA.2020.040.

[Cot21a] G. Cotti, *Degenerate Riemann–Hilbert–Birkhoff problems, semisimplicity, and convergence of WDVV-potentials*, Lett. Math. Phys. 111, 99 (2021).

[Cot21b] G. Cotti, *Riemann–Hilbert–Birkhoff inverse problem for semisimple flat F-manifolds, and convergence of oriented associativity potentials*, arXiv:2105.06329, 2021.

[CG17] G. Cotti and D. Guzzetti, *Analytic Geometry of Semisimple Coalescent Frobenius Structures*, Random Matrices Theory Appl., Vol. 6 (2017), no. 4, 1740004, 36 pp.

[CG18] G. Cotti and D. Guzzetti, *Results on the Extension of Isomonodromy Deformations with a Resonant Irregular Singularity*, Random Matrices Theory Appl., Vol. 07 (2018), no. 4, 1840003, 27 pp.

[DE95] J.W. Demmel and A. Edelman, *The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms*, Linear Algebra and its Applications, 230 (1995), pp. 61–87.

[DH21] L. David, C. Hertling, *Meromorphic connections over F-manifolds*, in "Integrability, Quantization, and Geometry. I. Integrable systems", Proceedings of Symposia in Pure Mathematics, Vol. 103, Editors S. Novikov, I. Krichever, O. Ogievetsky, S. Shlosman, Amer. Math. Soc., Providence, RI, 2021, 171–216, arXiv:1912.03331.

[dR54] G. de Rham, *Sur la division de formes et de courants par une forme linéaire*, Comment. Math. Helv., 28 (1954), 346–352.

[EEK97] A. Edelman, E. Elmroth, and B. Kågström, *A geometric approach to perturbation theory of matrices and and matrix pencils. Part I: Versal deformations*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 653–692.

[EEK99] A. Edelman, E. Elmroth, and B. Kågström, *A geometric approach to perturbation theory of matrices and and matrix pencils. Part II: a stratification-enhanced staircase algorithm*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 667–699.

[EJK03] E. Elmroth, P. Johansson, and B. Kågström, *Bounds for the distance between nearby Jordan and Kronecker structures in a closure hierarchy*, J. of Mathematical Sciences, 114 (2003), pp. 1765–1779.

[FG02] K. Fritzsche and H. Grauert, *From holomorphic functions to complex manifolds*, Springer, 2002.

[FMS21] C. Fevola, Y. Mandelshtam, B. Sturmfels, *Pencils of Quadrics: Old and New*, Le Matematiche, Vol. LXXVI, Issue II, pp. 319–335 (2021).

[FR66] O. Forster, K.J. Ramspott, *Okasche Paare von Garben nicht-abelscher Gruppen*, Invent. Math. 1, 260–286 (1966).

[For03] F. Forstnerič, *The Oka Principle for Multivalued Sections of Ramified Mappings*, Forum Math. 15(2), 309–328 (2003).

[For17] F. Forstnerič, *Stein Manifolds and Holomorphic Mappings*, 2nd edn. Springer, New York (2017).

[Gan59] F.R. Gantmakher, *The theory of matrices*, Chelsea, New York 1959.

[GLR06] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, SIAM (2006).

[GL09] I.C. Gohberg, J. Leiterer, *Holomorphic Operator Functions of One Variable and Applications*, Birkhäuser, Basel (2009).

[Gur88] R.M. Guralnick, *Similarity of holomorphic matrices*, Linear Algebra Appl. 99, 85–96 (1988).

[Guz18] D. Guzzetti, *Notes on Non-Generic Isomonodromy Deformations*, SIGMA 14 (2018), 087, 34 pages.

[Her21] C. Hertling, *Rank 2 Bundles with Meromorphic Connections with Poles of Poincaré Rank 1*, SIGMA 17 (2021), 082, 73 pages

[HP94a] W. Hodge, D. Pedoe, *Methods of Algebraic Geometry. Vol. I.* (Cambridge Mathematical Library). Cambridge: Cambridge University Press. (1994). doi:10.1017/CBO9780511623875.

[HP94b] W. Hodge, D. Pedoe, *Methods of Algebraic Geometry. Vol. II.* (Cambridge Mathematical Library). Cambridge: Cambridge University Press. (1994). doi:10.1017/CBO9780511623899.

[HR18] G. Hardy, S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proceedings of the London Mathematical Society, Second Series, 17, 75–115, (1918).

[HS66] P.-F. Hsieh and Y. Sibuya, *Note on regular perturbations of linear ordinary differential equations at irregular singular points*, Funkcial. Ekvac. 8 (1966), 99–108.

[Jac75] N. Jacobson, *Lectures in Abstract Algebra II. Linear Algebra*, Springer, New York (1975).

[Jan88] R. Janz, *Holomorphic Families of Subspaces of a Banach Space*, in: Arsene G. (eds) Special Classes of Linear Operators and Other Topics. Operator Theory: Advances and Applications, vol 28. Birkhäuser, Basel (1988).

[JMU81] M. Jimbo, T. Miwa and K. Ueno, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I*, Physica 2D (1981), p. 306–352.

[JLP76] W.B. Jurkat, D.A. Lutz, and A. Peyerimhoff, *Birkhoff invariants and effective calculations for meromorphic linear differential equations; Part I*, J. Math. Anal. Appl. 53 (1976), 438–470.

[Kab76] W. Kaballo, *Projektoren und relative Inversion holomorpher Semi-Fredholmfunktionen*, Math. Ann., 219 (1976), pp. 85–96.

[Kab12] W. Kaballo, *Meromorphic generalized inverses of operator functions*, Indagationes Mathematicae, Volume 23, Issue 4, Pages 970–994 (2012).

[Kan79] R. Kaneiwa, *An asymptotic formula for Cayley's double partition function* $p(2;n)$, Tokyo J. Math. 2(1), 137–158 (1979).

[Kan80] R. Kaneiwa, *Errata for "An asymptotic formula for Cayley's double partition function* $p(2;n)$*"*, Tokyo J. Math. 3(2), 461–461 (1980).

[Lei17] J. Leiterer, *On the Jordan structure of holomorphic matrices*, ArXiv e-prints (2017): arXiv:1703.09535.

[Lei20] J. Leiterer, *On the Similarity of Holomorphic Matrices*, J. Geom. Anal. 30, 2731–2757 (2020). ArXiv e-prints (2017): arXiv:1703.09530.

[Lor14] P. Lorenzoni, *Darboux–Egorov system, bi-flat* F*-manifolds and Painlevé VI*, IMRN Vol. 2014, No. 12, pp. 3279–3302.

[Mal83a] B. Malgrange, *Déformations de systèmes différentiels et microdifférentiels*. Séminaire E.N.S. Mathématique et Physique (L. Boutet de Monvel, A. Douady & J.-L. Verdier, eds.), Progress in Math., vol. 37, Birkhäuser, Basel, Boston, 1983, p. 351–379.

[Mal83b] B. Malgrange, *Sur les déformations isomonodromiques, II.* Séminaire E.N.S. Mathématique et Physique (L. Boutet de Monvel, A. Douady & J.-L. Verdier, eds.), Progress in Math., vol. 37, Birkhäuser, Basel, Boston, 1983, p. 427–438.

[Mal86] B. Malgrange, *Deformations of differential systems, II.* J. Ramanujan Math. Soc. **1** (1986), p. 3–15.

[Mas59] P. Masani, *On a result of G.D. Birkhoff on linear differential systems*, Proc. Amer. Math. Soc. 10 (1959), 696–698.

[NY04] D. Novikov, S. Yakovenko, *Lectures on meromorphic flat connections*, In book: Normal forms, bifurcations and finiteness problems in differential equations (pp.387-430), Publisher: Kluwer Acad. Publ, June 2004, DOI:10.1007/978-94-007-1025-2_11.

[Pet99] V.M. Petrogradsky, *Growth of finitely generated polynilpotent Lie algebras and groups, generalized partitions, and functions analytic in the unit circle*, Internat. J. Algebra Comput. 9 (2) (1999) 179–212.

[Pet00] V.M. Petrogradsky, *On growth of Lie algebras, generalized partitions, and analytic functions*, Discrete Mathematics, Vol. 217:1–3 (2000), 337–351.

[Sab98] C. Sabbah, *Frobenius manifolds: isomonodromic deformations and infinitesimal period mappings*, Expositiones Mathematicae 16 (1998), 1–58.

[Sab07] C. Sabbah, *Isomonodromic deformations and Frobenius manifolds*, Universitext, Springer & EDP Sciences (2007) (in French: 2002)

[Sab21] C. Sabbah, *Integrable deformations and degenerations of some irregular singularities*, Publ. RIMS Kyoto Univ., **57** (2021), no. 3–4, to appear, arXiv:1711.08514v3, 35 pages.

[Seg83] C. Segre, *Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni*, Mem. R. Acc. Scienze Torino, Vol. 36 (1883), p. 3–86. In: Corrado Segre, *Opere*, a cura della Unione Matematica Italiana, Volume III, Edizione Cremonese, Roma, 1961, p. 25–126.

[Seg12] C. Segre, *Mehrdimensionale Räume*, Enzyklopädie der mathematischen Wissenschaften (1912).

[Shu70] M.A. Shubin, *Holomorphic families of subspaces of a Banach space*, Integral Equations Operator Theory, 2 (1979), pp. 407–420 Translated from Mat. Issled. 5 (1970) 153–165 (in Russian)

[Sib62] Y. Sibuya, *Simplification of a system of linear ordinary differential equations about a singular point*, Funkcial. Ekvac. 4 (1962), 29–56.

[Sib90] Y. Sibuya, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Translations of Mathematical Monographs, vol. 82, American Math. Society, Providence, RI, 1990, (Japanese edition: 1976).

[Spa65] K. Spallek, *Differezierbare und holomorphe Funktionen auf analytischen Mengen*, Math. Ann. 161, 143–162 (1965).

[Spa67] K. Spallek, *Über Singularitäten analytischer Mengen*, Math. Ann. 172, 249–268 (1967).

[Syl51] J.J. Sylvester, *An enumeration of the contacts of lines and surfaces of the second order*, Phil. Mag. 1:2 (1851), 119–140.

[Thi78] Ph.G.A. Thijsse, *Decomposition theorems for finite meromorphic operator functions*, Thesis, Amsterdam, 1978.

[Thi85] Ph.G.A. Thijsse, *Global holomorphic similarity to a Jordan form*, Results Math. 8, 78–87 (1985).

[Tur63] H. Turrittin, *Reduction of ordinary differential equations to the Birkhoff canonical form*, Trans. Amer. Math. Soc. 107 (1963), p. 485–507.

[Was62] W. Wasow, *On Holomorphically Similar Matrices*, J. Math. Anal. Appl. 4, 202–206 (1962).

[Was65] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Pure and Applied Mathematics, Vol. XIV, Interscience Publishers John Wiley & Sons, Inc., New York-LondonSydney, 1965.

[Was85] W. Wasow, *Linear turning point theory*, Springer-Verlag New York (1985).